

78号院 105W

★ Function of one complex variable (OTM 11), J.B. Conway

Complex analytic, L Ahlfors

Complex analysis, Stein

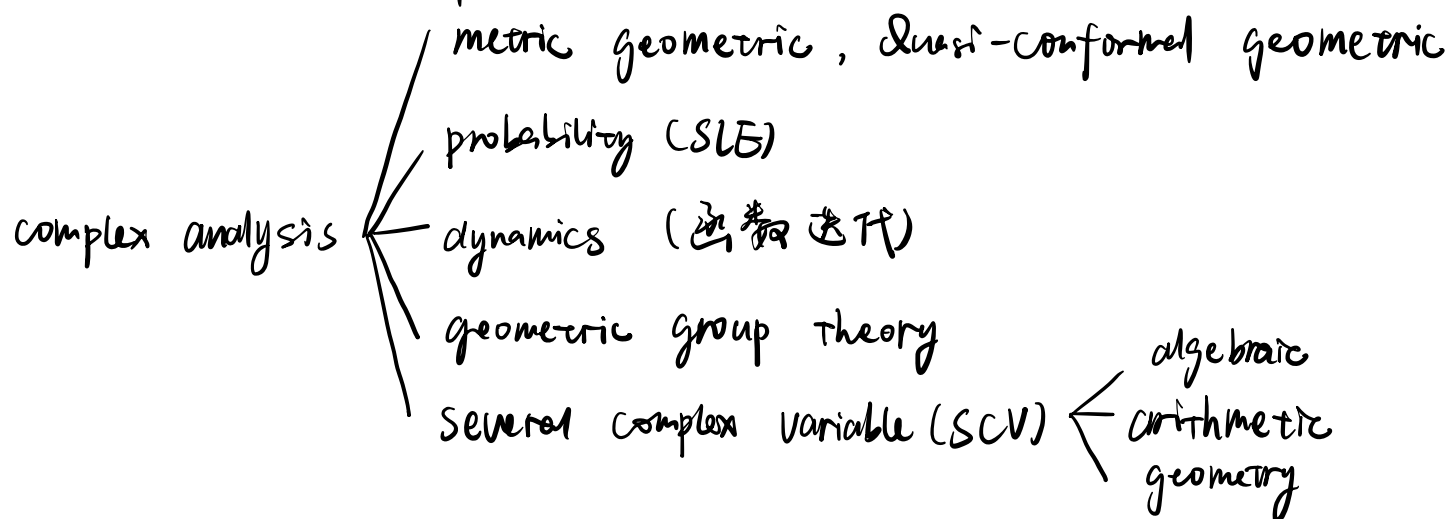
Real and complex analysis, Rudin

作业 30% + 期中 30% + 期末 40%

└ Bonus Questions (课上布置)

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x,y) = (u(x,y), v(x,y))$

Cauchy-Riemann Equation: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$



周四. 六 3-4 p.m. 77201 Seminar

Bonus Question: (Tropical Algebra)

Is $(\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$ a field? Identities?

Which axioms are missing?

交换 / 结合 / 分配 / identity (单位元)

一. 复数和复函数

1.1 $\mathbb{C}$ 的构造

$$\mathbb{C} := \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$$

$$(a, b) + (c, d) = (a+c, b+d)$$

$$(a, c) \times (c, d) = (ac-bd, ad+bc)$$

$\Rightarrow (\mathbb{C}, +, \times)$ is a field

$$\mathbb{R} \cong \{(a, 0) : a \in \mathbb{R}\} \subseteq \mathbb{C}$$

定义: $i = (0, 1)$ (imaginary unit)

$$z = (a, b) = a + ib, \quad a, b \in \mathbb{R}$$

$$\bar{z} = a - ib \quad (\text{complex conjugate})$$

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \quad (\text{absolute value / norm})$$

$$\text{Arg } z = \{\theta : z = |z| \cos \theta + i |z| \sin \theta\}$$

$$\arg(z) \in \text{Arg } z \cap [0, 2\pi) \quad (\text{principal argument})$$

$$\text{complex plane: } \mathbb{C} \cong \mathbb{R}^2$$

Triangle inequality: $|z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|$

Line: $ax + by + c = 0, \quad a, b, c \in \mathbb{R} \Leftrightarrow \bar{B}z + B\bar{z} + C = 0, \quad B = \frac{a+ib}{i}, \quad C \in \mathbb{R}$

Circle: $|z - B| = R > 0, \quad B \in \mathbb{C} \Leftrightarrow -z\bar{z} + B\bar{z} + \bar{B}z + C = 0, \quad C = -B\bar{B} + R^2, \quad B \in \mathbb{C}$

1.2 $\mathbb{C}$ 上的几何

• $\mathbb{C}$ 的度量 & 拓扑性质: $d(z_1, z_2) = |z_1 - z_2| \Rightarrow (\mathbb{C}, d)$ 为度量空间

• 一些记号:

$$A^\circ \text{ 或 } \text{int } A := \bigcup \{G: G \subseteq A, G \text{ 开}\}$$

$$\bar{A} := \bigcap \{F: F \subseteq A, F \text{ 闭}\}$$

$$\partial A = \bar{A} \cap \overline{(X \setminus A)}$$

• 定义: 称 (X, d) 连通, 若只有 $\emptyset$ 和 X 是既开又闭的.

证: 路径连通 $\Rightarrow$ 连通.

• 定理: $G \subseteq \mathbb{C}$ 开, λ :

G 连通 $\Leftrightarrow G$ 路径连通 $\Leftrightarrow \forall x, y \in G, x, y$ 可用有限段线性路径连通

• Cauchy Thm: (X, d) 完备 $\Leftrightarrow \forall$ 非空闭 $F_n, F_1 \supset F_2 \supset \dots, \text{diam}\{F_n\} \rightarrow 0,$
 $\exists c, \bigcap_{n=1}^{\infty} F_n = \{c\}$

• 性质: (1) A 开 $\Leftrightarrow A = \text{int } A$

(2) A 闭 $\Leftrightarrow A = \bar{A}$

(3) $\text{int } A = X \setminus \overline{(X \setminus A)}, \bar{A} = X \setminus \text{int}(X \setminus A), \partial A = \bar{A} \setminus \text{int } A$

(4) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(5) (X, d) 完备, $Y \subseteq X, \lambda$: (Y, d) 完备 $\Leftrightarrow Y$ 闭

- $\gamma: [0,1] \rightarrow \mathbb{C}$, $\gamma(t) = (x(t), y(t))$ 译: path 蕴含 γ 连续
 continuous curve: $\gamma \in C^0$
 smooth curve: $\gamma \in C^1$ 且 $\forall t \in [0,1], (x'(t))^2 + (y'(t))^2 \neq 0$
 piecewise smooth curve: $\exists 0 = x_0 < \dots < x_n = 1$, $\gamma|_{[x_{i-1}, x_i]}$ smooth curve
 rectifiable: $\int_0^1 \sqrt{x'(t)^2 + y'(t)^2} dt < \infty$
 区域: 路径连通开集
 Jordan curve: $\gamma: [0,1] \rightarrow \mathbb{C}$ 连续且 $\gamma(0) = \gamma(1)$ 且 $\gamma(x) \neq \gamma(y)$, $\forall 0 < x < y \leq 1$

• 定理: (Jordan curve Thm)

给定 $\mathbb{C}$ 中的 Jordan curve γ , $\gamma: [0,1] \rightarrow \mathbb{C}$ 恰有 2 个连通分支.

• Bonus Question: Find the original Jordan's "proof", and complete it.

• 定义: 区域 D 简单连通 (simply connected),

若 $\forall$ Jordan curve $\gamma \subseteq D$, $\exists$ 有界区域 D_0 , s.t. $\partial D_0 = \gamma$.

• 定义: $S \subset \mathbb{C}$, $f: S \rightarrow \mathbb{C}$, $z_0 \in \mathbb{C}$ 为一个 极限点,

若 $\forall \varepsilon > 0$, $B(z_0, \varepsilon) \cap S \neq \emptyset$.

• 扩充复平面: $\hat{\mathbb{C}}$ as the Riemann sphere via stereographic projection

译号 chordal metric: $dl(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}$, $z_1, z_2 \in \hat{\mathbb{C}}$

$dl(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$, $z \in \hat{\mathbb{C}}$

二. 解析 / 全纯函数

2.1 解析函数

- 定义: $A \subseteq \mathbb{C}$, 称 f 在 A 上解析, 若 f 在开集 $0 \ni A$ 上解析.
- 定义: 开集 $G \subseteq \mathbb{C}$, 解析函数 $f: G \rightarrow \mathbb{C}$ 为 单叶解析函数 (univalent function), 若 f 为单射.
此处不要求双射!
- 定义: f 为 Conformal mapping 共形映射, 若:
① 保角 ② $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$ 存在. (保方向)
- 定义: 开集 $G_1, G_2 \subseteq \mathbb{C}$, 解析函数 $f: G_1 \rightarrow G_2$ 为 双全纯函数 (biholomorphic function), 若 f 为 双射 且 f^{-1} 解析.
 f 双全纯 $\Rightarrow f$ 为 conformal mapping

定理: $f: G \rightarrow \mathbb{C}$ univalent, $\mathcal{M}$:

(1) $f'(z) \neq 0$

(R 中不恒: $y=x^3$)

(2) $f(G)$ 开.

反函数也解析 (R 中不一定)

(3) $f^{-1}: f(G) \rightarrow G$ 解析 且 $f^{-1}(f(z)) = \frac{1}{f'(z)}$

Cauchy-Riemann Equations: $f(x, y) = u(x, y) + i v(x, y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (*)$$

定理 1: f 在 $z_0 = u_0 + i v_0$ 可导, $\mathcal{M}$ z_0 处 (*) 成立.

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta x \rightarrow 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} = \frac{\partial u(x_0, y_0)}{\partial x} + i \frac{\partial v(x_0, y_0)}{\partial x}$$
$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \frac{f(z_0 + i \Delta y) - f(z_0)}{i \Delta y} = \frac{\partial v(x_0, y_0)}{\partial y} - i \frac{\partial u(x_0, y_0)}{\partial y}$$

· 定理 2: $u, v \in C^1(G, \mathbb{R})$, 则 $f = u + iv$ 在 G 上解析 $\Leftrightarrow$ (*) 在 G 成立

$\Rightarrow$: 用上面的定理

$$\Leftarrow: u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

$$= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) + u(x_0, y_0 + \Delta y) - u(x_0, y_0)$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) \Delta x + \frac{\partial u}{\partial y}(x_0, y_0) \Delta y + o(\sqrt{|\Delta x|^2 + |\Delta y|^2})$$

$$\text{同理 } v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$$

$$= \frac{\partial v}{\partial x}(x_0, y_0) \Delta x + \frac{\partial v}{\partial y}(x_0, y_0) \Delta y + o(\sqrt{|\Delta x|^2 + |\Delta y|^2})$$

$$\Rightarrow u \Delta z = \Delta x + i \Delta y$$

$$\text{则 } f(z_0 + \Delta z) - f(z_0) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)(x_0, y_0) \cdot \Delta x$$

$$+ \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right)(x_0, y_0) \cdot \Delta y + o(|\Delta z|)$$

$$\begin{aligned} \text{所以 CR} &= \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right)(x_0, y_0) \cdot \Delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial u}{\partial x} \right)(x_0, y_0) \cdot \Delta y + o(|\Delta z|) \\ &= \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right)(x_0, y_0) \cdot \Delta z + o(|\Delta z|) \end{aligned}$$

· 定理 3: 若 $u, v \in C^2(G, \mathbb{R})$ 且满足 (*), 则:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\text{从而 Laplacian: } \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ 同理 } \Delta v = 0$$

· 定义: $u \in C^2(G, \mathbb{R})$ 满足 $\Delta u = 0$, 则 u 称为 调和函数 (harmonic function)

$u, v: G \rightarrow \mathbb{R}$ harmonic, 称 v 为 u 的 共轭调和函数 (harmonic conjugate)

若 $f = u + iv$ 解析 $(\Leftrightarrow u, v$ 满足 (*))

注: (1) v 为 u 的 harmonic conjugate $\Leftrightarrow u$ 为 $-v$ 的 harmonic conjugate

(2) v_1, v_2 are both harmonic conjugates of u .

唯一性 $\mathcal{M}$ $V_1 - V_2$ 为 constant (on each connected component of G)

定理: $\mathcal{U} \subseteq \mathbb{C}$ simply connected region, 存在性

$\mathcal{M}$ $\forall$ harmonic $u: \mathcal{U} \rightarrow \mathbb{R}$ $\exists$ harmonic conjugate

取定 $z_0, p_0 = (x_0, y_0) \in \mathcal{U}$,

任取从 p_0 到 z 的分段线性曲线 γ_1, γ_2

$$\mathcal{M}: \int_{\gamma_1 - \gamma_2} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx = \iint_D (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) dx dy = 0$$

记 $v(z) = \int_{\gamma_1} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$, $\mathcal{M}$ $v(z)$ 与 γ_1 的选取无关

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u \in C^2 \Rightarrow \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

证: $u(x, y) = \ln(x^2 + y^2)$ 在 $\mathbb{C} \setminus \{0\}$ 上 harmonic, $\ln z$ 的多值性!

但设有 harmonic conjugate. 解微分方程可证明.

定理: $\mathcal{U}$ simply connected region, $u, v \in C^2(\mathcal{U}, \mathbb{R})$,

$f = u + iv \neq 0, \forall z$, 且 f 解析.

$\mathcal{M}: \exists$ 解析 $g: \mathcal{U} \rightarrow \mathbb{C}$, s.t. $e^{g(z)} = f(z), \forall z \in \mathcal{U}$.

取 $u(z) = \ln|f(z)|$, 可验证 $\Delta u(z) = 0$

$\Rightarrow \exists v$ harmonic, s.t. $h = u + iv$ 在 $\mathcal{U}$ 上解析.

$$\Rightarrow \left| \frac{e^{h(z)}}{f(z)} \right| = \left| \frac{e^{u(z)}}{f(z)} \right| = 1, \forall z \in \mathcal{U}$$

$\Rightarrow e^{h(z)} = f(z) e^{ic}, c \in \mathbb{R}$, 取 $g(z) = h(z) - ic$ 即可.

定理: $\mathcal{U}$ simply connected region, $u, v \in C^2(\mathcal{U}, \mathbb{R})$,

$f = u + iv \neq 0, \forall z$, 且 f 解析.

若 $0 \in \Omega \subset \mathbb{C}$, $\exists$ 解析 $h: \Omega \rightarrow \mathbb{C}$ s.t. $h'(z) = f(z), \forall z \in \Omega$.

取 g 使 $e^{g(z)} = f(z)$. 取 $h = e^{\frac{1}{2}g(z)}$ 即可.

· 定理: f 在区域 Ω 上解析, Jacobian of $f: (x, y) \mapsto (u, v)$ 为 $|f'(z)|^2$

$$J_f(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial v}{\partial x} \right|^2 = |f'(z)|^2$$

· 推论: $f: \Omega \rightarrow \mathbb{C}$ 解析, $f'(z) \in C(\Omega, \mathbb{C}), f'(z) \neq 0$,

则 $\exists z_0$ 的邻域 D , s.t.

(1) $f(D)$ 开

(2) $f: D \rightarrow f(D)$ 双射

反函数存在定理 $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

(3) $f^{-1}: f(D) \rightarrow D$ 解析, 且 $f^{-1}(f(z)) = \frac{1}{f'(z)}$

2.2 级数

· 定义: $a_n \in \mathbb{C}, n \in \mathbb{N}$, 则称 $\sum_{n=1}^{\infty} a_n$ 收敛到 z ,

$$\text{若 } \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m > N, \left| \sum_{n=1}^m a_n - z \right| < \varepsilon$$

称 $\sum a_n$ 绝对收敛 若 $\sum |a_n|$ 收敛.

· 命题: 收敛 $\Rightarrow$ 绝对收敛

· 定义: $\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf \{a_n, a_{n+1}, \dots\}, \quad \limsup_{n \rightarrow \infty} \{a_n, a_{n+1}, \dots\}$

· 定义: 函数项级数 $\sum f_n(z)$ 在 $S \subseteq \mathbb{C}$ 上 一致收敛 到 $g(z)$.

$$\text{即 } \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m > N, \forall z \in S, |g(z) - \sum_{n=1}^m f_n(z)| < \varepsilon$$

· 幂级数: $\sum_{n=0}^{\infty} a_n(z-a)^n$ 的收敛半径 R : $\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$

$|z-a| < R$: 绝对收敛

$|z-a| > R$: 发散

$\forall r \in (0, R)$, 在 $\bar{B}(a, r)$ 上一致收敛.

证: $\forall r \in (0, R)$, 记 $r_0 = \frac{R+r}{2}$

$$\text{则 } \exists N, \forall n > N, |a_n|^{\frac{1}{n}} < \frac{1}{r_0}$$

$$\Rightarrow \forall z \in \bar{B}(0, r), \forall n > N, |a_n z^n| < \left(\frac{|z|}{r_0}\right)^n \leq \left(\frac{r}{r_0}\right)^n < 1$$

$\Rightarrow$ 在 $\bar{B}(0, r)$ 上一致收敛且绝对收敛.

· 性质: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ 若存在则 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$

· 定义: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^x = \cos x + i \sin x$

· 命题: $\sum a_n, \sum b_n$ 绝对收敛, 则 $\sum c_n$ 绝对收敛, 其中 $c_n = \sum_{k=0}^n a_k b_{n-k}$

$$\text{且 } \sum c_n = (\sum a_n)(\sum b_n)$$

· 性质: $e^{z_1+z_2} = e^{z_1} e^{z_2}$

· 命题: $\sum a_n(z-a)^n, \sum b_n(z-a)^n$ 收敛半径 $\geq r > 0$. $c_n = \sum_{k=0}^n a_k b_{n-k}$

$$\mathcal{M}: \sum (a_n + b_n)(z-a)^n = \sum a_n(z-a)^n + \sum b_n(z-a)^n$$

$$\sum c_n(z-a)^n = \left(\sum a_n(z-a)^n \right) \left(\sum b_n(z-a)^n \right)$$

收敛半径 $\geq r$. $\forall z \in D(a, r)$.

命题: $f(z) = \sum a_n(z-a)^n$ 收敛半径 $R > 0$, $\mathcal{M}$:

$$(1) \forall k \geq 1, \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z-a)^{n-k} \text{ 收敛半径为 } R$$

$$(2) f \text{ 可导且 } f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z-a)^{n-k}$$

$$(3) a_n = \frac{1}{n!} f^{(n)}(a)$$

证: 只需证 $k=1$ 的情况. 取定 $r \in (0, R)$. $\forall z, z' \in D(a, r)$.

$$\begin{aligned} & \left| \frac{f(z) - f(z')}{z - z'} - \sum_{n=1}^{\infty} n a_n (z-a)^{n-1} \right| \\ & \leq \left| \sum_{n=0}^k \left(a_n \frac{(z-a)^n - (z'-a)^n}{z - z'} - n a_n (z-a)^{n-1} \right) \right| + \underbrace{\sum_{n=k+1}^{\infty} \left| a_n \frac{(z-a)^n - (z'-a)^n}{z - z'} \right|}_I + \underbrace{\sum_{n=k+1}^{\infty} |n a_n (z-a)^{n-1}|}_{II} \end{aligned}$$

$$\text{则 } |I| \leq |a_n| \sum_{n=k+1}^{\infty} |z-a|^n \cdot |z'-a|^n \leq |a_n| n r^{n-1} \quad I$$

$$\Rightarrow \sum_{n=k+1}^{\infty} |I| \leq \sum_{n=k+1}^{\infty} n |a_n| r^{n-1} \text{ 在 } r < R \text{ 时收敛.}$$

同理 $\sum_{n=k+1}^{\infty} |II|$ 收敛

$$\Rightarrow \forall \varepsilon > 0, \exists k, \sum_{n=k+1}^{\infty} |I| < \frac{\varepsilon}{3}, \sum_{n=k+1}^{\infty} |II| < \frac{\varepsilon}{3}.$$

$$\text{则 } \sum_{n=0}^k |I - II| \xrightarrow{z \rightarrow z'} 0$$

2.3 多值函数 (multivalued function)

定义: $F: S \rightarrow \mathcal{P}(\mathbb{C}) := \{\mathbb{C} \text{ 的子集}\}$ 称为 S 上 多值函数

证: $f: D \rightarrow \mathbb{C}$, $\mathcal{M} f^{-1}$ 是 $f(D)$ 上多值函数

定义: F 在区域 U 上单值, 若 $\exists$ 解析 $f: U \rightarrow \mathbb{C}$, $f(z) \in F(z)$.

则 f_n 称为 f 的一个 (解析)分支.

例: $f = e^z$

$$f^{-1} = \text{Ln}(z) : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{F}(\mathbb{C})$$

$$z \mapsto \text{Ln}(z) := \ln|z| + i \text{Arg}(z)$$

$$= \{ \ln|z| + i(\text{Arg}(z) + 2k\pi) : k \in \mathbb{Z} \}$$

定义: 连通 Hausdorff space R 称为 Riemann surface,

若 charts $\{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$ 使得:

(1) $U_\alpha \subseteq \mathbb{R}$ 开, $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha) \subseteq \mathbb{C}$ 是同胚.

$$(2) \bigcup_{\alpha \in \Lambda} U_\alpha = \mathbb{R}$$

(3) 若 $U_\alpha \cap U_\beta \neq \emptyset$, 则 $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ 是双全纯的.

定义: R 是 Riemann 面. atlas $\{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$

$f : \mathbb{R} \rightarrow \mathbb{C}$ 称为 holomorphic, 若 $f \circ \phi_\alpha^{-1}$ 在 $\phi_\alpha(U_\alpha)$ 上 holomorphic, $\forall \alpha$

定义: R_1, R_2 是 Riemann 面. atlas $\{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$ 及 $\{(V_\beta, \psi_\beta) : \beta \in \bar{\Lambda}\}$

$f : R_1 \rightarrow R_2$ 称为 holomorphic,

若 $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ 在 $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$ 上 holomorphic, $\forall \alpha, \beta$

例: $\hat{\mathbb{C}}$ 是 Riemann 面 ($\hat{\mathbb{C}}$ 是 Riemann 球)

$f(z) = z^n : \mathbb{C} \rightarrow \mathbb{C}$ 可 holomorphically 延拓到 holomorphic 函数 $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$

• Angle preserving:

$\gamma_1, \gamma_2: [-1, 1] \rightarrow \mathbb{C}$ 是 $\mathbb{C}^1$ smooth,

$$\gamma_1(0) = \gamma_2(0) = z$$

$$\theta = \arg \gamma_1'(0) - \arg \gamma_2'(0)$$

for $\gamma_1, f \circ \gamma_2: [-1, 1] \rightarrow \mathbb{C}$ 是 $\mathbb{C}^1$ smooth,

$$f(\gamma_1(0)) = f(\gamma_2(0)) = f(z)$$

$$\arg (f \circ \gamma_1)'(0) - \arg (f \circ \gamma_2)'(0)$$

$$= \arg (f'(z) \cdot \gamma_1'(0)) - \arg (f'(z) \cdot \gamma_2'(0))$$

$$= \arg (f'(z)) + \arg (\gamma_1'(0)) - \arg (f'(z)) - \arg (\gamma_2'(0))$$

$$= \arg (\gamma_1'(0)) - \arg (\gamma_2'(0)) = \theta$$



2.4 Möbius Transformation

定义: $f(z) = \frac{az+b}{cz+d}$ 称为 linear fractional transformation

$ad-bc \neq 0$ 时称为 Möbius transformation

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C}, ad-bc \neq 0 \right\}$$

• $\text{PSL}(2, \mathbb{C}) / \{\pm I\} \rightarrow \{\text{Möbius trans.}\} \cong \mathcal{M}$ 是群同构

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{az+b}{cz+d}$$

• $\mathcal{M} = \langle \begin{matrix} z \mapsto ze^{i\theta} & z \mapsto az & z \mapsto \frac{1}{z} & z \mapsto z+a \\ \text{rotations,} & \text{dilation,} & \text{inversion,} & \text{translation} \end{matrix} \rangle$

• 交比: $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ 互不相同. $(z, z_2, z_3, z_4) := \frac{z-z_3}{z-z_4} / \frac{z_2-z_3}{z_2-z_4}, z \in \hat{\mathbb{C}}$

$$z_2 \mapsto 1, z_3 \mapsto 0, z_4 \mapsto \infty$$

• Möbius 变换保交比: $(z, z_2, z_3, z_4) = (T(z), T(z_2), T(z_3), T(z_4)), \forall T \in \mathcal{M}, z \in \hat{\mathbb{C}}$
 $\triangleq S(z)$

$$ST^{-1}(T(z_2)) = 1, ST^{-1}(T(z_3)) = 0, ST^{-1}(T(z_4)) = \infty$$

$$\Rightarrow ST^{-1}(z) = (z, T(z_2), T(z_3), T(z_4))$$

$$\Rightarrow S(z) = (T(z), T(z_2), T(z_3), T(z_4))$$

• 命题: $z_2, z_3, z_4 \in \mathbb{C}$ 互不同, w_2, w_3, w_4 互不同,

$$\exists! f \in \mathcal{M}, f(z_i) = w_i, i=2,3,4.$$

• 命题: z_1, z_2, z_3, z_4 共圆 $\Leftrightarrow (z_1, z_2, z_3, z_4) \in \mathbb{R}$.

• 定义: P 为过 z_2, z_3, z_4 的圆. z, z^* 关于 P 对称,

$$\text{若 } (z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \quad (\text{与 } z_2, z_3, z_4 \text{ 的选取无关})$$

• 命题: $S \in \mathcal{M}$, P 圆, z, z^* 关于 P 对称, $\forall S(z), S(z^*)$ 关于 $S(P)$ 对称.

$$\begin{aligned} (S(z), S(z_2), S(z_3), S(z_4)) &= (z, z_2, z_3, z_4) \\ &= \overline{(z^*, z_2, z_3, z_4)} \\ &= \overline{(S(z^*), S(z_2), S(z_3), S(z_4))} \end{aligned}$$

三. Complex Integration (参考 Conway 第4章)

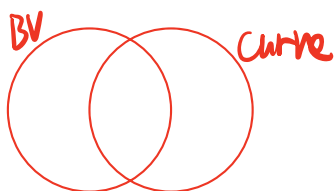
3.1 Riemann-Stieltjes Integral

• 定义: $\gamma: [a, b] \rightarrow \mathbb{C}$ 称为有界变差 (bounded variation),

若 $\exists M > 0$, $\forall$ 分划 $P: a = t_0 < \dots < t_m = b$,

$$V(\gamma; P) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

total variation: $V(\gamma) = \sup \{V(\gamma; P) : P \text{ 有限分划}\}.$



定义中未规定 γ 是连续或是 curve!

注: $\exists$ 连续曲线不是 BV 的.

命题: $\gamma: [a, b] \rightarrow \mathbb{C}$ 是 BV 的, 则:

(1) 分划 $P \subseteq Q$, 则 $V(\gamma; P) \leq V(\gamma; Q)$

(2) $\beta: [a, b] \rightarrow \mathbb{C}$ 是 BV 的, 则 $\alpha\gamma + \beta\delta$ 也是 BV 的.

$$\text{且 } V(\alpha\gamma + \beta\delta) \leq \alpha V(\gamma) + \beta V(\delta)$$

命题: $\gamma: [a, b] \rightarrow \mathbb{C}$ 分段 C^1 , 则 $V(\gamma) = \int_a^b |\gamma'(t)| dt \Rightarrow \gamma \in BV$

定理: $\gamma: [a, b] \rightarrow \mathbb{C}$ 是 BV 的, $f: [a, b] \rightarrow \mathbb{C}$ 连续.

则 $\exists I \in \mathbb{C}$, $\forall \varepsilon > 0$, $\exists \delta > 0$,

$\forall$ 分划 $P = \{t_0, \dots, t_m\}$ 使 $\max |t_k - t_{k+1}| < \delta$

$$\left| I - \sum_{k=1}^m f(z_k) (\gamma(t_k) - \gamma(t_{k-1})) \right| \leq \varepsilon, \quad \forall z_k \in [t_{k-1}, t_k]$$

定义: $I := \int_a^b f(t) d\gamma(t)$ 称为 Riemann-Stieltjes 积分

命题: $f, g \in C[a, b]$, $\gamma, \delta \in BV[a, b]$, 则 $\forall \alpha, \beta \in \mathbb{C}$,

(1) $\int_a^b (\alpha f + \beta g) d\gamma = \alpha \int_a^b f d\gamma + \beta \int_a^b g d\gamma$

$$(2) \int_a^b f d(\alpha \gamma \oplus \beta \delta) = \alpha \int_a^b f d\gamma + \beta \int_a^b f d\delta$$

concatenation: "曲线加法":  首尾相接

命题: $\gamma \in BV[a, b]$, $f \in C[a, b]$, $a = t_0 < t_1 < \dots < t_n = b$,

$$\gamma: \int_a^b f d\gamma = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f d\gamma|_{[t_{k-1}, t_k]}$$

定理: γ 分段 C^1 , $f \in C[a, b]$, $\gamma: \int_a^b f d\gamma = \int_a^b f(\gamma(t)) \gamma'(t) dt$
R.S. 积分 Riemann 积分

分实部、虚部考虑, 用微分中值定理.

定义: 称 $\gamma: [a, b] \rightarrow \mathbb{C}$ 可求长, 若 $\gamma \in BV[a, b]$.

$\gamma: [a, b] \rightarrow \mathbb{C}$ 可求长, $f: \gamma([a, b]) \rightarrow \mathbb{C}$ 连续,

$$\gamma \text{ 线积分}: \int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

例: (1) $\gamma = e^{it}$, $t \in [0, 2\pi]$, $f(z) = \frac{1}{z}$.

$$\int_{\gamma} \frac{1}{z} dz = \int_0^{2\pi} e^{-it} i e^{it} dt = 2\pi i \text{ 常用}$$

(2) $\gamma = e^{it}$, $t \in [0, 2\pi]$, $f(z) = z^m$

$$\begin{aligned} \int_{\gamma} z^m dz &= \int_0^{2\pi} e^{imt} (i e^{it}) dt = i \int_0^{2\pi} e^{i(m+1)t} dt \\ &= i \int_0^{2\pi} \cos((m+1)t) dt - \int_0^{2\pi} \sin((m+1)t) dt = 0 \end{aligned}$$

命题: $\gamma \in BV[a, b]$, $\varphi: [c, d] \rightarrow [a, b]$ 单调连续, $\varphi(c) = a$, $\varphi(d) = b$,

则 $\gamma \circ \varphi \in BV[c, d]$, 且 $\forall f \in C(\gamma([a, b]))$,

$$\int_{\gamma} f dz = \int_{\gamma \circ \varphi} f dz \quad \text{参数替换下不变.}$$

• 命题: γ 可求长, f 在 γ 上连续, 则:

$$(1) \int_{\gamma} f dz = - \int_{-\gamma} f dz \quad (\Gamma\gamma)(t) = \gamma(a+b-t)$$

$$(2) \left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| dz \leq V(\gamma) \cdot \sup |f|$$

• 定理: $G \subseteq \mathbb{C}$ 开, $\gamma: [a, b] \rightarrow G$ 可求长,

若 $F, G \in C(G)$, $F'(w) = f(w)$, $\forall w \in G$.

$$\gamma \int_{\gamma} f dz = F(\beta) - F(\alpha) \quad \text{N-L 公式}$$

$$\begin{aligned} \textcircled{1} \gamma \text{ 分段 } C', \quad \int_{\gamma} f dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b (F \circ \gamma)'(t) dt \\ &= F(\gamma(b)) - F(\gamma(a)) \end{aligned}$$

$\textcircled{2} \gamma$ 可求长, 用分段 C' 的 γ_n 逼近.

3.2 幂级数与解析函数.

• 定理: $\gamma \in BV[a, b]$, 连续 $f_n: \gamma \rightarrow \mathbb{C}$, $n \in \mathbb{N}$, $f: \gamma \rightarrow \mathbb{C}$, $f_n \Rightarrow f$,

$$\gamma \int_{\gamma} f_n dz = \int_{\gamma} f dz \quad \text{一致连续} \rightarrow \int \text{和 } \lim \text{可交换.}$$

• Recall: Green 公式: $\Omega \subseteq \mathbb{C}$ 有界. $\partial\Omega$ 分 (有限) 段 C'

$u(x, y), v(x, y)$ 在 Ω 的某开邻域 C'

$$\gamma: \int_{\partial\Omega} u dx + v dy = \iint_{\Omega} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$\text{从而: } \int_{\partial\Omega} f(z) dz = \int_{\partial\Omega} (u + iv)(dx + i dy)$$

$$= \iint_{\Omega} \left(\left(-\frac{\partial}{\partial y} + i \frac{\partial}{\partial x} \right) u + \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) v \right) dx dy$$

$$= 2i \iint_{\Omega} \frac{\partial}{\partial \bar{z}} f dz dx dy$$

• Cauchy 定理: $\Omega \subset \mathbb{C}$ 有界区域, $\partial\Omega$ 分(有限)段 C^1

$f: \Omega \rightarrow \mathbb{C}$ 连续且 f 在 Ω 上解析. 则: $\int_{\partial\Omega} f(z) dz = 0$

引理: $D \subset \mathbb{C}$ 为三角形区域, f 在 D 的某开邻域解析.

$$\text{则: } \int_{\partial D} f(z) dz = 0$$

若 $\int_{\partial D} f(z) dz = M \neq 0$.

$$\Rightarrow \int_{\partial D} f(z) dz = \sum_{j=1}^4 \int_{\partial D_j} f(z) dz = M$$

$$\Rightarrow \exists j, \left| \int_{\partial D_j} f(z) dz \right| \geq \frac{|M|}{4}$$

对 D_j 同理做剖分, 得 $\Delta^{(1)} \supset \Delta^{(2)} \supset \dots$, 满足 $\left| \int_{\partial \Delta^{(k)}} f dz \right| \geq \frac{|M|}{4^k}$

$$\Rightarrow \exists z_0, \bigcap \Delta^{(k)} = \{z_0\}$$

$$\text{即 } f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0) \quad (z \rightarrow z_0)$$

$$\Rightarrow \frac{|M|}{4^k} \leq \left| \int_{\partial \Delta^{(k)}} f dz \right| = \left| \int_{\partial \Delta^{(k)}} o(z - z_0) dz \right|$$

$$\leq \max_{z \in \Delta^{(k)}} \left| \frac{o(z - z_0)}{z - z_0} \right| \cdot \int_{\partial \Delta^{(k)}} |z - z_0| dz$$

$$\leq \max_{z \in \Delta^{(k)}} \left| \frac{o(z - z_0)}{z - z_0} \right| \cdot \text{diam} \Delta^{(k)} \cdot l(\partial \Delta^{(k)})$$

$$= \max_{z \in \Delta^{(k)}} \left| \frac{o(z - z_0)}{z - z_0} \right| \cdot \frac{\text{diam}(D)}{2^k} \cdot \frac{l(\partial D)}{2^k}$$

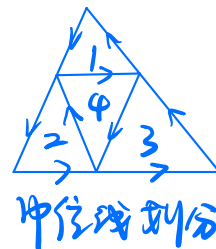
$$\Rightarrow |M| \leq \max_{z \in \Delta^{(k)}} \left| \frac{o(z - z_0)}{z - z_0} \right| \cdot \text{diam}(D) \cdot l(\partial D)$$

令 $k \rightarrow \infty$ 知矛盾!

回到原题. Ω 有有限条直线段为边界的多边形 D , $D \subset \Omega$,

$$\text{且 } \left| \int_{\partial\Omega} f(z) dz - \int_{\partial D} f(z) dz \right| < \varepsilon$$

对 D 三角剖分即可.



• Cauchy 积分公式: Ω 有界区域, $\partial\Omega$ 分(有限)段 C^1 ,

$f \in C(\bar{\Omega})$ 且在 Ω 上解析.

$$21: \forall z \in D, f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw$$

$$\text{取 } B(z, \varepsilon) \subseteq D.$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial D \setminus B(z, \varepsilon)} \frac{f(w)}{w-z} dw = 0$$

$$\begin{aligned} \Rightarrow \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \int_{B(z, \varepsilon)} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{f(z) + (w-z)f'(z) + o(|w-z|)}{w-z} dw \\ &= f(z) + \frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{o(|w-z|)}{w-z} dw \end{aligned}$$

$$\text{令 } \varepsilon \rightarrow 0^+ \text{ 即可.}$$

· 定理: f 在 $B(z_0, R)$ 解析, 则 $f(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n$, $z \in B(z_0, R)$
其中 $a_n = \frac{1}{n!} f^{(n)}(z_0)$ 且 $\sum_{n=0}^{+\infty} a_n (z-z_0)^n$ 收敛半径 $\geq R$

$$\text{取定 } r \in (0, R)$$

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dz, \quad z \in B(z_0, r)$$

$$\text{即 } \frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{w-z_0}{w-z} = \frac{1}{w-z_0} \sum_{n=0}^{+\infty} \left(\frac{z-z_0}{w-z_0} \right)^n$$

$$\Rightarrow f(z) = \sum_{n=0}^{+\infty} \left(\frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w) dw}{(w-z_0)^{n+1}} \right) (z-z_0)^n$$

$$\text{故 } a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w) dw}{(w-z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0)$$

证: 反之也成立 解析 $\Leftrightarrow$ 可展成幂级数.

· 推论: (1) $f: G \rightarrow \mathbb{C}$ 解析 $\Rightarrow f$ 光滑.

(2) $f: G \rightarrow \mathbb{C}$ 解析, $B(z_0, r) \subseteq G$

$$\Rightarrow f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

(3) (Cauchy Estimate) $f \in B(z_0, R)$ 解析,

$$|f(z)| \leq M, \quad \forall z \in B(z_0, R), \quad \text{则 } |f^{(n)}(z_0)| \leq \frac{n! M}{R^n}$$

3.3 解析函数的零点

· 定义: $f: G \rightarrow \mathbb{C}$ 解析, $a \in G$, $f(a)=0$,

称 a 为 f 的 m 阶零点, 若于解析函数 $g: G \rightarrow \mathbb{C}$,

s.t. $f(z) = (z-a)^m g(z)$, $\forall z \in G$ 且 $g(a) \neq 0$

· 定义: 称解析函数 $f: \mathbb{C} \rightarrow \mathbb{C}$ 为 整函数.

· 命题: f 为整函数, 则 $f(z) = \sum_{n=0}^{\infty} a_n z^n$, 收敛半径 $R = +\infty$

· Liouville's Thm: f 为有界整函数, 则 $f \equiv C$

$$|f(z)| \leq \frac{M}{R}, \forall z \in \mathbb{C}, \forall R > 0$$

· 代数学基本定理: $p(z)$ 非常值多项式, 则 $\exists a \in \mathbb{C}$, $p(a)=0$

反证: 若无零点 $\Rightarrow f(z) = \frac{1}{p(z)}$ 为整函数

易见 $\lim_{z \rightarrow \infty} p(z) = \infty \Rightarrow \lim_{z \rightarrow \infty} f(z) = 0 \Rightarrow f$ 有界 $\Rightarrow f \equiv C$ 矛盾.

· 定理: 区域 Ω , $f: \Omega \rightarrow \mathbb{C}$ 解析, 以下等价:

(1) $f \equiv 0$

(2) $\exists z \in \Omega$, s.t. $\forall n \in \mathbb{N}_0$, $f^{(n)}(z) = 0$

(3) $Z := \{z \in \Omega: f(z) = 0\}$ 有 Ω 中聚点.

(1) $\Rightarrow$ (3): $\checkmark$

(3) $\Rightarrow$ (2): a 为 Z 的聚点, 则 $f(a) = 0$.

$$\text{若 } 0 = f(a) = f'(a) = \dots = f^{(n)}(a) \neq f^{(n+1)}(a).$$

$$\lambda) f(z) = \sum_{k=n+1}^{+\infty} a_k (z-a)^k, \quad z \in B(a, r) \subseteq \mathbb{C}.$$

$$\text{令 } g(z) := \sum_{k=n+1}^{\infty} a_k (z-a)^{k-n} \text{ 在 } B(a, r) \text{ 上解析. } g(a) = a_{n+1} \neq 0$$

$$\text{故 } \exists \varepsilon > 0, \quad g(z) \neq 0, \quad \forall z \in B(a, \varepsilon)$$

$$\text{由 } a \text{ 孤立 } \Rightarrow \exists b \in B(a, \varepsilon), \quad b \neq a, \quad g(b) = (b-a)^{n+1} f(b) = 0. \text{ 矛盾.}$$

$$(2) \Rightarrow (1): \text{ 记 } S = \{z \in \mathbb{C} : f^{(n)}(z) = 0, \forall n \in \mathbb{N}\} \text{ 易见 } S \text{ 闭}$$

$$\text{设 } a \in S, \quad f(z) = \sum_{n=0}^{+\infty} a_n (z-a)^n = 0, \quad z \in B(a, r)$$

$$\Rightarrow B(a, r) \subseteq \mathbb{C} \Rightarrow S = \mathbb{C} \Rightarrow S = \mathbb{C}$$

· 推论: f, g 在区域 Ω 内解析, $\lambda) f \equiv g \Leftrightarrow \{z : f(z) = g(z)\}$ 有聚点

· 推论: f 在区域 Ω 内解析, $f \neq 0$, $\lambda) \forall a \in \Omega$, 若 $f(a) = 0$,

$$\text{则 } \exists n \in \mathbb{N}, \exists g : \Omega \rightarrow \mathbb{C} \text{ 解析, s.t. } g(a) \neq 0, \quad f(z) = g(z) (z-a)^n$$

$$g(z) := \begin{cases} \frac{1}{n!} f^{(n)}(z), & z = a \\ \frac{1}{(z-a)^n} f(z), & z \in \Omega \setminus \{a\}. \end{cases}$$

$$\text{其中 } n \text{ 满足: } f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0 \neq f^{(n)}(a)$$

$$\exists r, \quad f(z) = \sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k, \quad z \in B(a, r).$$

$$\text{易见 } g(z) = \sum_{k=0}^{\infty} \frac{f^{(k+n)}(a)}{k!} (z-a)^k, \quad z \in B(a, r) \Rightarrow g \text{ 解析.}$$

· 推论: 开集上非常值解析函数的零点是孤立的.

· Maximum Modulus Thm (最大模原理): $f : \Omega \rightarrow \mathbb{C}$ 解析, Ω 区域.

$$\text{满足: } \exists a, \quad \forall z \in \Omega, \quad |f(a)| \geq |f(z)|, \quad \lambda) f \equiv c.$$

- Open mapping property: $f: \mathbb{C} \rightarrow \mathbb{C}$ 解析非常值, $\mathbb{C}$ 开集, 则 $\forall$ 开集 $U \subset G$, $f(U)$ 开.

证 of MM7: 若 f 非常值 $\Rightarrow \exists r, f(B(a, r))$ 开. 证:

3.4 闭曲线环绕数

- 命题: $\gamma: [0, 1] \rightarrow \mathbb{C}$ 是可求长闭曲线, $a \notin \gamma$, 则 $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} \in \mathbb{Z}$

不妨 $\gamma \in C^1$, $g: [0, 1] \rightarrow \mathbb{C}$, $g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a} ds$

则 $g(0) = 0$, $g(1) = \int_{\gamma} \frac{1}{z-a} dz$, $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$, $\forall t \in [0, 1]$.

易验证: $\frac{d}{dt}(e^{-g(t)}(\gamma(t)-a)) = e^{-g} \gamma' - g' e^{-g}(\gamma-a) = 0$

$$\Rightarrow e^{-g(0)}(\gamma(0)-a) = \gamma(0)-a = \gamma(1)-a = e^{-g(1)}(\gamma(1)-a)$$

$$\Rightarrow e^{-g(1)} = 1 \Rightarrow \exists k \in \mathbb{Z}, g(1) = 2\pi i k$$

- 定义: $\gamma: [0, 1] \rightarrow \mathbb{C}$ 是可求长闭曲线, 则对 $a \notin \gamma$, 定义 γ 关于 a 的环绕数

$$n(\gamma; a) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a} dz$$

- 性质: (1) $n(\gamma; a) = -n(-\gamma; a)$, $\forall a \notin \gamma$.

$$(2) n(\gamma + \delta; a) = n(\gamma; a) + n(\delta; a), \forall a \notin \gamma \cup \delta$$

(3) $n(\gamma; a)$ 在 $\mathbb{C} \setminus \gamma$ 的每个连通分支上为常值. 连通分支上连续.

(4) $n(\gamma; a) = 0$, 若 $a \in \mathbb{C} \setminus \gamma$ 的无界连通分支

$$a \rightarrow +\infty, |n(\gamma; a)| \leq \frac{1}{2\pi} V(\gamma) \frac{1}{|a|} \rightarrow 0, \text{ 其中 } V(\gamma) \text{ 表示长度.}$$

3.5 Cauchy 定理 & 积分公式

• Lemma: 给定 γ 可求长, $\gamma: \gamma \rightarrow \mathbb{C}$ 连续, $m \in \mathbb{N}$,

$$M \quad F_m(z) = \int_{\gamma} \frac{\gamma(w)}{(w-z)^m} dw, \quad z \in \mathbb{C} \setminus \gamma \text{ 的补集.}$$

$$\& F'_m(z) = m F_{m+1}(z)$$

Step 1: 先证 $F_m \in C(\mathbb{C} \setminus \gamma)$. 取定 $a \in \mathbb{C} \setminus \gamma$.

$$\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} = (z-a) \left(\frac{1}{(w-z)^m(w-a)} + \dots + \frac{1}{(w-z)(w-a)^m} \right)$$

$$\text{Step 2: } \frac{F_m(z) - F_m(a)}{z-a} = \int_{\gamma} \frac{\gamma(w)(w-a)^{-1}}{(w-z)^m} dw + \dots + \int_{\gamma} \frac{\gamma(w)(w-a)^{-m}}{w-z} dw$$

$$\begin{aligned} \Rightarrow F'_m(a) &= \lim_{z \rightarrow a} \int_{\gamma} \frac{\gamma(w)(w-a)^{-1}}{(w-z)^m} dw + \dots + \lim_{z \rightarrow a} \int_{\gamma} \frac{\gamma(w)(w-a)^{-m}}{w-z} dw \\ &= \int_{\gamma} \frac{\gamma(w)(w-a)^{-1}}{(w-a)^m} dw + \dots + \int_{\gamma} \frac{\gamma(w)(w-a)^{-m}}{w-a} dw \\ &= m F_{m+1}(a). \quad \checkmark \end{aligned}$$

• Cauchy's Integral formula (I):

$G \subseteq \mathbb{C}$ 开, $f: G \rightarrow \mathbb{C}$ 解析, $\gamma \in G$ 为可求长闭曲线,

$$\& n(\gamma; w) = 0, \quad \forall w \in \mathbb{C} \setminus G.$$

$$M: \forall a \in G \setminus \gamma, \quad n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz$$

$$\text{pf: 定义 } \varphi: G \times G \rightarrow \mathbb{C}, \quad \varphi(z, w) = \begin{cases} \frac{f(z) - f(w)}{z-w}, & z \neq w \\ f'(z), & z = w \end{cases}$$

易验证: $\varphi \in C(G \times G)$, $z \mapsto \varphi(z, w)$ 是解析的, $\forall w$

$$\text{定义 } g: G \rightarrow \mathbb{C}, \quad g(z) = \int_{\gamma} \varphi(z, w) dw, \quad z \in G \rightarrow \text{开集} \uparrow$$

$$\int_{\gamma} \frac{f(w)}{w-z} dw, \quad \text{若 } \underline{n(\gamma; a) = 0}$$

$$\text{由 } n(\gamma; a) = 0 \text{ 时, } \int_{\gamma} \varphi(z, w) dw = \int_{\gamma} \frac{f(w) - f(z)}{w-z} dw$$

$$= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z) n(\gamma; z) \cdot 2\pi i = \int_{\gamma} \frac{f(w)}{w-z} dw$$

故 g 为良定的解析函数. 用 Lemma

即 $|z| \rightarrow \infty$ 时, $z \in \mathbb{C} \setminus G$, $\int \frac{f(w)}{w-z} dw \rightarrow 0$ f 在 V 上有界.

$$\Rightarrow g \text{ 有界} \Rightarrow g \equiv 0$$

$$\Rightarrow \forall a \in G - V, \quad 0 = \int \frac{f(z) - f(a)}{z-a} dz = \int \frac{f(z)}{z-a} dz - f(a) n(V; a)$$

• Cauchy's Integral formula (II):

$G \subseteq \mathbb{C}$ 开, $f: G \rightarrow \mathbb{C}$ 解析, $V_1, \dots, V_m \subseteq G$ 为可求长闭曲线,

$$\text{且 } n(V_1; w) + \dots + n(V_m; w) = 0, \quad \forall w \in \mathbb{C} \setminus G$$

$$\text{则: } \forall a \in G \setminus \bigcup_{j=1}^m V_j, \quad f(a) \sum_{j=1}^m n(V_j; a) = \sum_{j=1}^m \frac{1}{2\pi i} \int_{V_j} \frac{f(z)}{z-a} dz$$

证法类似.

• Cauchy 定理: $f: G \rightarrow \mathbb{C}$, $V_1, \dots, V_m \subseteq G$, 则 $\sum_{k=1}^m \int_{V_k} f(z) dz = 0$
对 $f(z)(z-a)$ 用 Cauchy 积分公式 I.

• 定理: $f: G \rightarrow \mathbb{C}$, $V_1, \dots, V_m \subseteq G$, 则 $\forall k \in \mathbb{N}, \forall a \in G \setminus \bigcup_{k=1}^m V_k$,
 $f^{(k)}(a) \sum_{j=1}^m n(V_j; a) = k! \sum_{j=1}^m \frac{1}{2\pi i} \int_{V_j} \frac{f(z) dz}{(z-a)^{k+1}}$

证 $F_k(a) = \int_{V_j} \frac{f(z)}{(z-a)^{k+1}} dz$, 则 $F_k'(a) = k F_{k+1}(a)$. 归纳即可.

• 定理: $\mathcal{U}$ 区域, $f: \mathcal{U} \rightarrow \mathbb{C}$ 解析, 零点 $a_1, \dots, a_m$, $V \subseteq G$ 为可求长闭曲线,
 $a_i \notin V$, $n(V; w) = 0, \quad \forall w \in \mathbb{C} \setminus \mathcal{U}$, 则 $\frac{1}{2\pi i} \int_V \frac{f(z)}{f(z)} dz = \sum_{k=1}^m n(V; a_k)$

$$f(z) = (z-a_1) \dots (z-a_m) g(z), \quad g \text{ 解析且 } g \neq 0$$

$$\text{则: } \frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \dots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

由 Cauchy 定理: $\checkmark$

· 推论: $\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)-a} dz = \sum_{k=1}^m n(r; a_k)$, $a_1, \dots, a_m$ 为 $f(z)-a$ 的零点.

· 定理: $f: B(a, R) \rightarrow \mathbb{C}$ 解析, $\alpha = f(a)$, 蕴含 $B(\alpha, \delta) \subset f(B(a, \varepsilon))$
若 $f(z)-\alpha$ 有一个 m 阶零点 a 则 $\exists \varepsilon > 0, \exists \delta > 0$, ↑

局部形如 $\forall \beta \in B(\alpha, \delta) \setminus \{\alpha\}$, $f(z)-\beta$ 在 $B(a, \varepsilon)$ 上恰有 m 个一阶零点.

$f = z^m$ $\exists \varepsilon \in (0, \frac{1}{2}R)$, $\forall z \in B(a, 2\varepsilon) \setminus \{a\}$, $f(z) \neq \alpha$ 且 $f'(z) \neq 0$.
由零点的孤立性
取 $\gamma = a + \varepsilon e^{2\pi i t}$, $t \in [0, 1]$.

$$\phi = (f \circ \gamma)(t), \text{ 则 } \alpha \notin \phi$$

$\Rightarrow \exists \delta > 0, B(\alpha, \delta) \cap \phi = \emptyset \Rightarrow B(\alpha, \delta)$ 落在 $\mathbb{C}-\phi$ 的某连通分支内.

$$\begin{aligned} \text{则 } \forall \beta \in B(\alpha, \delta), m = n(\phi; \alpha) &= n(\phi; \beta) = \frac{1}{2\pi i} \int_{\phi} \frac{dw}{w-\beta} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)-\beta} = \sum_{k=1}^p n(r; z_k(\beta)) = p \end{aligned}$$

$\Rightarrow p = m$, 其中 $f(z_k(\beta)) = \beta$, $z_k(\beta) \in B(a, \varepsilon)$

而 $f'(z) \neq 0 \Rightarrow f(z_k(\beta))$ 为一阶零点.

· Pf of Open Mapping Thm:

$$\forall a \in U, \exists \varepsilon > 0, \delta > 0, B(f(a), \delta) \subseteq f(B(a, \varepsilon))$$

· 推论: $U \subset \mathbb{C}$, $f: U \rightarrow \mathbb{C}$ 单叶解析, $f(U) = U'$

则 $\forall z \in U$, $f'(z) \neq 0$, $f^{-1}: U' \rightarrow \mathbb{C}$ 解析, 且 $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$

若 $f'(z_0) = 0$, $f(z)-f(z_0)$ 有 m 阶零点, $m \geq 2$. 易知 f 单叶

f 开映射 $\Rightarrow f^{-1}$ 连续.

$z = f^{-1}(f(z))$. 利用复合求导法证. (Conway B9 Prop 2.20)

• Weierstrass 定理: f 非常值整函数, $\mathcal{A}$ $f(\mathbb{C})$ 在 $\mathbb{C}$ 稠密.

反证: $\exists z_0 \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$. $\mathcal{A} \exists r, B(z_0, r) \subset \mathbb{C} \setminus \overline{f(\mathbb{C})}$.

$g(z) = \frac{1}{f(z) - z_0}$ 整函数, $|g(z_0)| \leq \frac{1}{r} \Rightarrow g \equiv 0$ 矛盾.

• Little Picard Thm: f 整函数, $\mathcal{A} \#(\mathbb{C} \setminus f(\mathbb{C})) \leq 1$ 证明略.

• Mean Value Thm: $\mathcal{A} z \in \mathbb{R}^n, f: \mathbb{R}^n \rightarrow \mathbb{C}$ 解析.

$\forall z_0 \in \mathbb{R}^n, \forall r > 0$ 使 $\overline{B(z_0, r)} \subseteq \mathbb{R}^n$.

$$\text{有: } f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

• 推论: (i) $|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta$

$$(ii) |f(z_0)| \leq \frac{1}{\pi r^2} \int_{B(z_0, r)} |f(z)| ds$$

• Schwarz's Lemma: $f: B(0, 1) \rightarrow B(0, 1)$ 解析, $f(0) = 0, \mathcal{A}$:

(i) $\forall z \in B(0, 1), |f(z)| \leq |z|, |f'(0)| \leq 1$.

(ii) 以下等价: (a) $\exists z_0 \neq 0, |f(z_0)| = |z_0|$.

(b) $|f'(0)| = 1$.

(c) $f(z) = ze^{i\theta}, \theta \in [0, 2\pi)$

(i) $g(z) = \frac{f(z)}{z} = a_1 + a_2 z + a_3 z^2 + \dots$ 在 $B(0, 1)$ 解析.

边界取到最大模

MMP $\Rightarrow \forall r \in (0, 1), \forall z \in B(0, r), |g(z)| = \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}$

令 $r \rightarrow 1 \Rightarrow |g(z)| \leq 1 \Rightarrow |f(z)| \leq |z|, |f'(0)| = |g(0)| \leq 1$.

(ii) 若 $\exists z_0, \left| \frac{f(z_0)}{z_0} \right| = 1$, $\mathcal{A}$ 由 MMP: $\frac{f(z)}{z} = 0$ 常值.

· 定理: $g: B(0,1) \rightarrow B(0,1)$ 双全纯,

$$\text{则 } g(z) = e^{i\theta} \frac{z-z_0}{1-\bar{z}_0 z}, \quad z_0 \in B(0,1), \quad \theta \in [0, 2\pi)$$

$$\text{而 } l_{z_0}(z) = \frac{z-z_0}{1-\bar{z}_0 z}, \quad \text{则 } l_{z_0} \in |z| < |z_0|^{-1} \text{ 解析.}$$

$$\text{而 } l_{z_0} \circ l_{-z_0}(z) = z \Rightarrow l_{z_0} \text{ 双全纯.}$$

$$\text{取 } l = l_{g(0)}, \text{ 则 } l(g(0)) = 0,$$

$$\text{取 } f = l \circ g(z), \text{ 则 } f(0) = 0 \text{ 且 } f \text{ 双全纯.}$$

$$\Rightarrow |f'(0)| \leq 1 \text{ 且 } |(f^{-1})'(0)| \leq 1$$

$$\Rightarrow |f'(0)| = 1 \Rightarrow f(z) = e^{i\alpha} z = \frac{g(z) - g(0)}{1 - \overline{g(0)} g(z)} \quad \checkmark$$

$$\text{注: } \underline{A(B(0,1))} = \left\{ e^{i\theta} \frac{z-z_0}{1-\bar{z}_0 z} : \theta \in [0, 2\pi], z_0 \in B(0,1) \right\}$$

为 $B(0,1)$ 上 双全纯函数群.

· 定理: (Schwarz's Lemma 的一般情形) $f: B(0,1) \rightarrow B(0,1)$ 解析.

$$\forall z_1, z_2 \in B(0,1), \quad \forall \left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$$

$$\text{取等} \Leftrightarrow f \in A(B(0,1))$$

· 推论: $\forall z \in B(0,1), \quad \frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \quad \text{取等} \Leftrightarrow f \in A(B(0,1))$

$$z_2 = z, \quad \text{令 } z_1 \rightarrow z$$

· 定义: $B(0,1)$ 上定义 Poincaré 度量: $ds = \frac{|dz|}{1-|z|^2}$

$$l_p(\gamma) = \int_\gamma ds = \int_a^b \frac{|d\gamma(t)|}{1-|\gamma(t)|^2}$$

$$\text{若 } w \in A(B(0,1)), \quad \text{则 } \frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2}$$