

FIXED POINT PROPERTIES OF GROUP ACTIONS ON CAT(0) SPACES

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ABSTRACT. We give an expository account of fixed point properties of groups acting on CAT(0) spaces. We begin with Serre’s study on groups acting on trees, which classified groups that must act with a fixed point on a tree. We then introduce Farb and Bridson’s higher-dimensional analogues for actions on n -dimensional CAT(0) spaces. In particular, we demonstrate the results on reflection groups, arithmetic groups, mapping class groups, and braid groups.

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1. INTRODUCTION

Serre [Ser03] studied group actions on trees and proved that $\mathrm{SL}_3(\mathbb{Z})$ has *property FA*: any $\mathrm{SL}_3(\mathbb{Z})$ -action on a tree must have a fixed point. Later, Culler and Vogtmann [CV96] extended the results to mapping class groups and braid groups. They proved that $\mathrm{Mod}(S_g)$ has property FA when $g \geq 2$, and $B_n/Z(B_n)$ has property FA when $n \geq 5$. Since trees are typical examples of 1-dimensional CAT(0) spaces, it is natural to consider the high dimensional analogue of the above results. Farb [Far09] considered a group Γ acting on CAT(0) spaces and introduced the

corresponding *property* FA_n : any Γ -action on a n -dimensional complete $\text{CAT}(0)$ space must have a fixed point. By using Helly's theorem and its generalized form, Farb [Far09] proved the property FA_n for S -arithmetic groups and Chevalley groups. In particular, he proved that a Euclidean or hyperbolic reflection group about a n -simplex has property FA_{n-1} , and that $\text{SL}_n(\mathbb{Z})$ has property FA_{n-2} . Then Bridson [Bri10] proved that $\text{Mod}(\Sigma_g)$ has property FA_{g-1} when $g \geq 3$. He further gave a generalized scheme for fixed points of group actions on $\text{CAT}(0)$ spaces in [Bri25], called *Ample Duplication Criterion*, which argues that a group action must have a fixed point if it has a large number of mutually commuting conjugate subgroups that act with fixed points. He then applied this criterion to mapping class groups and braid groups.

In this paper, we give proofs of the results mentioned above; some of which are adapted for simplicity in this exposition. We start from studying group actions on trees in Section 2. We begin with the classical fixed-point-free action of $\text{SL}_2(\mathbb{Z})$ on a tree, and then investigate its group structure in 2.1 to elucidate why such an action exists. We prove in 2.2 the necessary and sufficient conditions for a group to have property FA given by Serre in [Ser03]. On the other hand, we prove that $\text{SL}_3(\mathbb{Z})$ must act with a fixed point on a tree in Theorem 2.21. We start with a single automorphism of trees, and proceed with nilpotent groups in 2.4. In 2.5, we finally prove that $\text{SL}_3(\mathbb{Z})$ has property FA by making use of its nilpotent subgroups. Using the same techniques, we also provide proofs of property FA of mapping class groups and braid groups modulo center in 2.6.

In Section 3, we use the same strategy to prove similar results on group actions on $\text{CAT}(0)$ spaces. After analyzing isometries of $\text{CAT}(0)$ spaces in 3.2 and studying the case of nilpotent groups in 3.3, we illustrate how fixed points of subgroups result in a global fixed point. While some results are listed in 3.4, we also introduce a more general theorem on intersections of subspaces called *Helly's theorem* in 3.5. We then provide a simple proof of property FA_n of reflection groups and arithmetic groups using Helly's theorem and Farb's idea [Far09] in 3.6. Finally, we prove Bridson's generalized version of Helly's theorem [Bri25], called *Bootstrap lemma*, in 3.7. Using this lemma, we can obtain Bridson's *Ample Duplication Criterion* [Bri25], as shown in 3.8. We conclude this paper by demonstrating how to use such techniques to prove that $\text{Mod}(\Sigma_g)$ has property FA_{g-1} , as well as a similar result on braid groups in 3.9.

2. GROUPS ACTING ON TREES

Recall that a *tree* is an undirected graph that is connected and has no cycles. Trees are among the simplest spaces on which groups can act, and their geometry strongly constrains such actions. A fundamental result about groups acting on trees is the theorem below

Theorem 2.1. *A group acts on a tree freely if and only if it is a free group.*

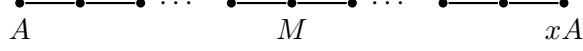
The idea used here is that a relation among the generators of a group G results in a loop in the graph which G acts on; see more details in Proposition 15 and Theorem 4 in [Ser03].

Since only a free group admits a free action on trees, it is natural to weaken the conditions and ask what groups can act on a tree without a fixed point. Note that when considering group actions on trees, we usually assume the action is without inversions, that is, no element swaps the endpoints of an edge. This is because an inversion always fixes the midpoint of the edge, but we usually take a tree as a combinatorial object and thus only the fixed vertices matter. In this paper, we always assume that a group acts on a tree without inversion.

Now the simplest example we may look at is the case of $\mathbb{Z}/2\mathbb{Z} = \langle x \rangle$.

Example 2.2. We will see below that every $\mathbb{Z}/2\mathbb{Z}$ -action on a tree has a fixed point. We choose an arbitrary vertex A . If the action has no fixed point, then $A \neq xA$. Let $[A, xA]$ be the unique

geodesic connecting A and xA . Since x interchanges A and xA , it reverses the segment $[A, xA]$ and must therefore fix its midpoint.



If the midpoint is a vertex M , then x fixes M . If otherwise the midpoint is the midpoint of an edge, then x must act as an inversion, which contradicts our assumption above.

On the other hand, some groups admit an action on a tree without any fixed point. Besides free groups, another example that one may be familiar with is the case of $\mathrm{SL}_2(\mathbb{Z})$.

Example 2.3. Recall that

$$\mathrm{SL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = I, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 \right\rangle.$$

We can construct a tree on which $\mathrm{SL}_2(\mathbb{Z})$ acts without fixed points; see Figure 1. In this tree, each red vertex is connected to two blue vertices, and each blue vertex is connected to three red vertices. Note that this tree is infinite. The generator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts as a 180° rotation fixing the degree-2 vertex A , while the generator $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ acts as a 120° rotation fixing the degree-3 vertex B . Hence this action does not have a global fixed point. The tree we constructed in Figure 1 is actually called the *Bass-Serre tree* for $\mathrm{SL}_2(\mathbb{Z})$. One may refer to [Bas93] to learn more about *Bass-Serre theory*.

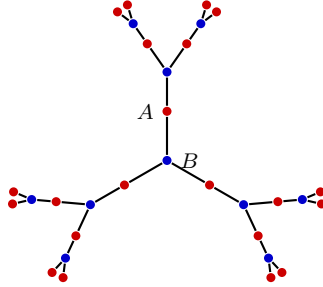


FIGURE 1. The Bass-Serre tree for $\mathrm{SL}_2(\mathbb{Z})$.

But what essentially makes $\mathrm{SL}_2(\mathbb{Z})$ admit a fixed-point-free action? If we look further into the group structure of $\mathrm{SL}_2(\mathbb{Z})$, we may notice that it has a splitting in some sense. Since

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}/4\mathbb{Z}, \quad \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}/6\mathbb{Z}, \quad \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 \right\rangle \cong \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 \right\rangle \cong \mathbb{Z}/2\mathbb{Z},$$

$\mathrm{SL}_2(\mathbb{Z})$ can be regarded as $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ glued together via $\mathbb{Z}/2\mathbb{Z}$. In the next section, we will call this kind of structure *an amalgam*, and construct a fixed-point free action for amalgams.

2.1. Amalgams. In this subsection, we define *amalgams* and construct a fixed-point-free group action on a tree for each amalgam.

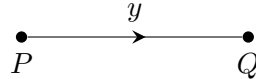
Definition 2.4 (Amalgam). Let A be a group, $(G_i)_{i \in I}$ a family of groups, and $\phi_i: A \rightarrow G_i$ an injective homomorphism for each $i \in I$. The quotient of the free product $*_{i \in I} G_i$ by the normal subgroup generated by $\{\phi_i(a)\phi_j(a)^{-1} \mid a \in A, i, j \in I\}$ is denoted by $*_A G_i$, and is called the *sum of the G_i with A amalgamated* (or simply the *amalgam* of the G_i over A). We also say that a group G is an *amalgam* if it can be written $G \simeq G_1 *_A G_2$ with A properly embedded in both G_1 and G_2 .

Example 2.5. $\mathrm{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$

We now consider the actions of amalgams on trees. Recall that in the case of $\mathrm{SL}_2(\mathbb{Z})$, the stabilizers of vertices are nontrivial. In fact, the stabilizer of A is $\mathbb{Z}/4\mathbb{Z}$, the stabilizer of B is $\mathbb{Z}/6\mathbb{Z}$, and the stabilizer of the edge connecting A and B is $\mathbb{Z}/2\mathbb{Z}$. For a general amalgam $*_A G_i$, we can construct a tree similarly so that the vertex stabilizers are G_i , and the edge stabilizers are A . This naturally leads to the concept of a *graph of groups*, a graph labeled with some groups for every vertex and edge.

Definition 2.6 (Graph of Groups, Definition 8 in [Ser03]). A *graph of groups* (G, T) consists of an oriented graph T , a group G_P for each $P \in \mathrm{vert} T$ (the vertex set of graph T), and a group G_y for each $y \in \mathrm{edge} T$ (the edge set of graph T), together with a monomorphism $G_y \rightarrow G_{t(y)}$ (denoted by $a \mapsto a^y$), where $t(y)$ denotes the terminal vertex of y . One requires in addition that $G_{\bar{y}} = G_y$, where $\bar{y}$ denotes the inverse edge y . If T is a tree, then (G, T) is called a *tree of groups*. We let $G_T = \varinjlim (G, T)$ denote the direct limit of the tree of groups (G, T) , which is called the amalgam of the G_P along the G_y .

Example 2.7. Take T to be a segment PQ , as shown below. Given three groups G_P , G_Q , and $G_y = G_{\bar{y}}$, together with two monomorphisms $G_{\bar{y}} = G_y \rightarrow G_P$ and $G_y \rightarrow G_Q$. The group G_T is equal to $G_P *_{G_y} G_Q$.



Now, in analogy of the case of $\mathrm{SL}_2(\mathbb{Z})$, we construct a group action for amalgams generally.

Theorem 2.8 (Graph Associated with a Tree of Groups, Theorem 9 in [Ser03]). *Let (G, T) be a tree of groups. Then there exists a graph X (called the graph associated with (G, T)) containing T and an action of $G_T = \varinjlim (G, T)$ on X which is characterized (up to isomorphism) by the property that T is a fundamental domain for $X \bmod G_T$, and the stabilizer of each $P \in \mathrm{vert} T$ (resp. $y \in \mathrm{edge} T$) in G_T is G_P (resp. G_y). Moreover, X is a tree.*

Sketch of proof. Construct X with vertices $\coprod_P G_T/G_P$ and edges $\coprod_y G_T/G_y$. To show X is a tree, we first reduce to the case where T is finite by taking direct limits over finite subtrees. We then proceed by induction on $n = |\mathrm{vert} T|$. For the induction step ($n > 1$), pick a terminal vertex $P = t(y)$ and set $T' = T \setminus \{P\}$, which gives $G_T = G_{T'} *_{G_y} G_P$. By the induction hypothesis, $X' = G_{T'} T'$, the $G_{T'}$ -orbit of T' , is a tree. Let $\tilde{X}$ be the graph derived from X by contracting each of the gX' to a point. G_T acts on $\tilde{X}$ with fundamental domain $T/T' = (T') \xrightarrow{y} P$; the stabilizers of (T') , P and y are respectively $G_{T'}$, G_P and G_y . Then $\tilde{X}$ is a tree, and hence so is X . $\square$

In particular, the tree X constructed in Theorem 2.8 has the property that an amalgam act on it without global fixed points. Conversely, given a group action on a tree whose fundamental domain is a tree, we can prove that the group is actually an amalgam of the G_P along the G_y , where G_P and G_y denote the vertex stabilizer and the edge stabilizer respectively. More precisely, we have the following:

Theorem 2.9 (Characterization of Group Actions on Trees, Theorem 10 in [Ser03]). *Let G be a group acting on a graph X with a tree $T \subseteq X$ as a fundamental domain. Let (G, T) be the tree of groups whose G_P and G_y are the stabilizers in G of $P \in \text{vert } T$ and $y \in \text{edge } T$ (with inclusions $G_y \rightarrow G_{t(y)}$ as monomorphisms). Let $G_T = \varinjlim (G, T)$. Let $G_T \rightarrow G$ be the homomorphism induced by the inclusions $G_P \rightarrow G$, and let $\tilde{X}$ be the tree associated with (G, T) , equipped with the unique equivariant map $\tilde{X} \rightarrow X$ extended by the identity map $T \rightarrow T$. Then the following properties are equivalent:*

- (1) X is a tree.
- (2) $\tilde{X} \rightarrow X$ is an isomorphism.
- (3) $G_T \rightarrow G$ is an isomorphism.

The idea of proof is to make use of the constructions of the two maps above, and to apply the fact that if $f: \tilde{X} \rightarrow X$ be a locally injective morphism of a connected graph $\tilde{X}$ into a tree X , then f is injective. For a more detailed proof, see [Ser03].

From the two theorems above, we can see that any tree of groups can be realized through a group action of an amalgam on a tree. In fact, we can get more information of the structure of groups acting on trees; see [Ser03]. The following theorem explains how a group action on a tree decomposes the group into vertex stabilizers together with a free quotient coming from the quotient graph.

Theorem 2.10 (Corollary 1 of Theorem 13 in [Ser03]). *Let G be a group acting without inversion on a nonempty tree X . Let R be the subgroup of G generated by the vertex stabilizers G_P for all $P \in \text{vert } X$. Then R is a normal subgroup of G , and G/R can be identified with the fundamental group of the graph $Y = G \backslash X$.*

2.2. Property FA. In this subsection, we define *property FA* and give the necessary and sufficient conditions for it.

Definition 2.11 (Property FA, [Ser03]). A group G is said to have *property FA* if there is a global fixed point for every action of G on a tree (without inversion).

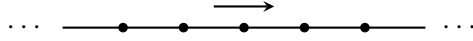
We know from the previous subsection that a group with property FA cannot be an amalgam. In fact, we only need two more conditions to ensure that a group has property FA. More precisely:

Theorem 2.12 (Theorem 15 in [Ser03]). *Suppose that G is a denumerable group. Then G has property FA if and only if the following three conditions are satisfied:*

- (1) G is not an amalgam;
- (2) G has no quotient isomorphic to $\mathbb{Z}$;
- (3) G is finitely generated.

Proof. (FA) $\Rightarrow$ (1): If G is an amalgam $G_1 *_A G_2$, as we proved in Theorem 2.8, there is a tree X on which G acts, with a segment PQ as fundamental domain, the stabilizers of P, Q being G_1, G_2 respectively. Since G is distinct from G_1 and G_2 , we then have $X^G = \emptyset$, which contradicts (FA).

(FA) $\Rightarrow$ (2): If G has a quotient isomorphic to $\mathbb{Z}$, we can make it act by translations on the doubly infinite chain



and this contradicts (FA).

(FA) $\Rightarrow$ (3): Since G is denumerable, it is the union of an increasing sequence $G_1 \subset G_2 \subset \dots \subset G_n \subset \dots$ of finitely generated subgroups. We construct a graph X whose vertex set is $\coprod_n G/G_n$, two vertices being joined by an edge if and only if they belong to two consecutive sets G/G_n and G/G_{n+1} and correspond under the canonical map $G/G_n \rightarrow G/G_{n+1}$. One checks that X is a tree, and G acts in an obvious way on X . Then there is a vertex P of X invariant under G . If $P \in G/G_n$ this implies that $G_n = G$, so G is finitely generated.

Conversely, suppose that G has properties (1), (2) and (3) and that it acts on a tree X . If $T = G \backslash X$, by Theorem 2.10, the fundamental group $\pi_1(T)$ of the graph T is isomorphic to a quotient of G . Since $\pi_1(T)$ is a free group, by (2), we must have $\pi_1(T) = \{1\}$. Therefore, T is a tree.

By Theorem 2.9, G can be identified with the group $G_T = \varinjlim (G, T)$, limit of the tree of groups defined by the groups G_P and G_y which fix the vertices P and edges y of T . It follows that G is the union of the groups $G_{T'} = \varinjlim (G, T')$ where T' runs through the set of *finite* subtrees of T .

Since G is finitely generated, there is a T' such that $G = G_{T'}$; we choose a minimal T' with this property. If T' reduces to a single vertex P we have $G = G_P$ and G has a fixed point. If not, T' has a terminal vertex P , and $T'' = T' - \{P\}$ is a tree. Let y denote the unique edge which joins P to T'' , then $G = G_{T'} = G_{T''} *_A G_P$, where $A = G_y$. By the minimality hypothesis on T' , we have $G_{T''} \neq G$ and $G_P \neq G$, which shows that G is an amalgam and contradicts hypothesis (1). $\square$

Although this criterion gives an equivalent condition of property FA, we will not use this to directly check whether a group has property FA, because it is difficult to determine whether a group is an amalgam. Instead, we develop another method to prove property FA by first analyzing the action of a single group element, then obtaining fixed points of nilpotent subgroups, and finally turning them into global fixed points.

We now record several elementary facts about property FA. See [Ser03] for a detailed proof.

Proposition 2.13. *Let G be a group.*

- (1) *Every finitely generated torsion group has property FA.*
- (2) *If G has property FA, then so does every quotient of G .*
- (3) *Let H be a normal subgroup of G . If H and G/H have property FA, then so does G .*
- (4) *Let H be a subgroup of finite index in G . If H has property FA, then so does G .*

Remark 2.14. Note that if a group G has property FA, a subgroup of finite index in G does not necessarily have property FA. For example, the Schwarz group $G = \langle a, b \mid a^A = b^B = (ab)^C = 1 \rangle$ (with integers $A, B, C \geq 2$) satisfies property FA, but when $1/A + 1/B + 1/C \leq 1$, it contains a finite-index subgroup H isomorphic to the fundamental group of a compact orientable surface of genus $g \geq 1$, which maps onto $\mathbb{Z}$ and thus does not satisfy property FA. See [Ser03] for more details.

2.3. Automorphisms of trees. In this subsection, we turn from the Bass-Serre construction to fixed point criteria. The goal is to prove property FA by checking fixed point behavior of certain generators and subgroups.

We first introduce some classic results on fixed points of an automorphism of a tree. The following proposition classifies the automorphisms of trees. See [Ser03] for a detailed proof.

Proposition 2.15. *Let s be an automorphism of a tree X . Then s has no fixed points if and only if there is a straight path in X stable under s , on which s induces a translation of non-zero amplitude.*

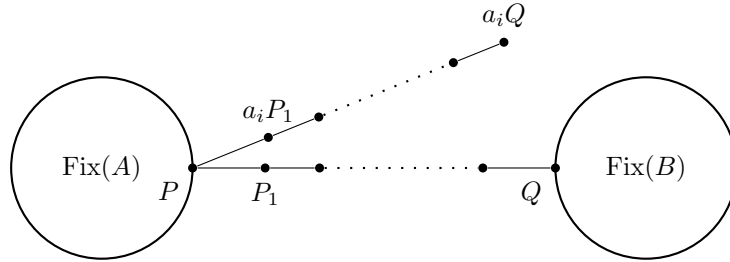
Sketch of proof. The “if” part is direct. To prove the “only if” part, suppose that s has no fixed points in X . Let $d(\cdot, \cdot)$ denote the standard graph metric on X . Let $m = \inf_{P \in \text{vert } X} d(P, sP)$, and

$S = \{P \in \text{vert } X \mid d(P, sP) = m\}$. Choose an arbitrary vertex $x_0 \in S$. Let $\gamma = [x_0, s(x_0)]$ be the unique geodesic path with vertices $x_0 = v_0, v_1, \dots, v_n = s(x_0)$. One can check that $v_i \in S$, for $i = 0, 1, \dots, n$. Consequently, the union $L = \bigcup_{k \in \mathbb{Z}} s^k(\gamma)$ forms a straight path in X that is stable under s and on which s induces a translation of non-zero amplitude $n > 0$. $\square$

Once we know how the generators act on a tree, we can ask when a group action has a global fixed point given that all generators have their respective fixed points. We start by considering two subgroups which generate the whole group.

Proposition 2.16 (Proposition 26 in [Ser03]). *Let G be a group generated by elements a_i, b_j and let A, B be the subgroups of G generated by the a_i and b_j respectively. Assume that G acts on a tree X so that A and B have a fixed point respectively, and so that for each pair (i, j) the automorphism $a_i b_j$ has a fixed point. Then G has a fixed point.*

Proof. Note that $\text{Fix}(G) = \text{Fix}(A) \cap \text{Fix}(B)$. Suppose that the trees $\text{Fix}(A)$ and $\text{Fix}(B)$ are disjoint. Let $[P, Q]$ be the geodesic joining them, where $P \in \text{vert } \text{Fix}(A)$, $Q \in \text{vert } \text{Fix}(B)$. Let P_1 be the vertex of $[P, Q]$ at distance 1 from P . We have $P_1 \notin \text{Fix}(A)$ and there is an index i such that $a_i P_1 \neq P_1$. Then P is the midpoint of the geodesic $[Q, a_i Q]$. But we have $a_i Q = a_i b_j Q$ for all j , and $a_i b_j$ has a fixed point, so the midpoint P of $[Q, a_i b_j Q]$ is fixed by $a_i b_j$. We then have $b_j P = a_i^{-1} P = P$, which means that P is fixed by all the b_j , contrary to the fact that $P \notin \text{Fix}(B)$. $\square$



If we apply the proposition to a finite number of generators, we have the following:

Corollary 2.17 (Corollary 2 of Proposition 26 in [Ser03]). *Suppose that G is generated by a finite number of elements $s_1, \dots, s_m$ such that the s_j and the $s_i s_j$ have fixed points. Then G has a fixed point.*

The results above reduce global fixed-point questions to checking fixed points for suitable generators and their products. In applications, however, it is often difficult to check these conditions directly for the whole group. Next, we introduce an intermediate step – we consider finitely generated nilpotent subgroups acting on trees. We show that the action either has a global fixed point, or is through translations on a single line. This restriction gives useful criteria for proving property FA for groups with large nilpotent subgroups.

2.4. Nilpotent groups. Based on the simple structure of nilpotent groups, we can determine whether a group action has fixed points simply by the action of its generators. We first give the following in analogy of Proposition 2.15.

Theorem 2.18 (Proposition 27 in [Ser03]). *Let G be a finitely generated nilpotent group acting on a tree X . Then either*

- (1) *there is a straight path T stable under G on which G acts by translations by means of a non-trivial homomorphism $G \rightarrow \mathbb{Z}$, or*
- (2) *$G \cdot x = x$ for some $x \in X$.*

Proof. We assume that G has no fixed points and prove that the statement in (1) holds. Since G is finitely generated and nilpotent, we can choose a composition sequence $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$, such that the successive quotients G_i/G_{i-1} are cyclic. We now proceed by induction on n .

The case $n = 0$ is trivial. Suppose then that $n \geq 1$ and apply the induction hypothesis to the group $H = G_{n-1}$. If H has a fixed point, then $\text{Fix}(H)$ is a non-empty G -invariant subtree of X , because H is a normal subgroup of G . Since H acts trivially on $\text{Fix}(H)$, the action of G on $\text{Fix}(H)$ factors through the quotient G/H . Since G has no fixed points, G/H acts without fixed points on $\text{Fix}(H)$. Thus, G/H must act by a non-trivial translation along a straight path T in $\text{Fix}(H)$. Then T is also stable under G , and G acts on T by translations via the canonical homomorphism $G \rightarrow G/H \rightarrow \mathbb{Z}$. If H has no fixed point, the straight path T stable under H is stable under G , because H is a normal subgroup of G . The isometry group of an infinite path T is $\text{Aut}(T) \cong D_\infty$, the infinite dihedral group. Its orientation preserving subgroup is $\mathbb{Z}$, the translations. So the action of G given by a homomorphism $G \rightarrow \text{Aut}(T)$ either contains a reflection and is infinite dihedral or its image is contained in $\mathbb{Z}$. But the former case is impossible, since the infinite dihedral group is not nilpotent. Therefore only the second case remains, as required. $\square$

Now we use the theorem above to understand global fixed points of actions of finitely generated nilpotent groups through the action of every generator.

Corollary 2.19. *Let G be a finitely generated nilpotent group acting on a tree. If G is generated by elements which have fixed points, then G has a fixed point.*

Proof. If G has no fixed points, by Theorem 2.18, at least one of the generators has a nontrivial image in $\mathbb{Z}$. But this element cannot have a fixed point, a contradiction. $\square$

In the case where the group G acts by translations along a path, we prove in the following corollary that its commutator subgroup must act trivially.

Corollary 2.20. *Let G be a finitely generated nilpotent group acting on a tree. Let G' be the commutator subgroup of G , and let s be an element of G such that $s^n \in G'$ for some integer $n \geq 1$. Then s has a fixed point.*

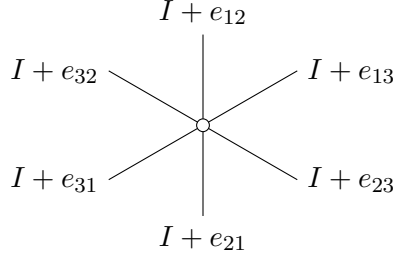
Proof. This is clear if G has a fixed point. If not, we have the homomorphism $G \rightarrow \mathbb{Z}$ in Theorem 2.18. The hypothesis on s implies that its image is zero. Therefore s leaves the straight path T fixed pointwise. $\square$

The two corollaries above are the main fixed point criteria used in proving property FA of $\text{SL}_3(\mathbb{Z})$, mapping class groups, and braid groups modulo center.

2.5. The case of $\text{SL}_3(\mathbb{Z})$. In contrast to the case of $\text{SL}_2(\mathbb{Z})$, we apply the results in the previous subsections to prove that $\text{SL}_3(\mathbb{Z})$ has property FA.

Theorem 2.21 (Theorem 16 in [Ser03]). *The group $\text{SL}_3(\mathbb{Z})$ has property (FA).*

Proof. The idea of proof is to make use of the nilpotent subgroups of $\text{SL}_3(\mathbb{Z})$ to yield fixed points for some specific group elements, and then apply Corollary 2.17 to yield a global fixed point. Let $i, j \in \{1, 2, 3\}$, and e_{ij} denote the elementary matrix whose entries are zero with the exception of a 1 in the i th row and j th column. It is well known that $\text{SL}_3(\mathbb{Z})$ is generated by $I + e_{ij}$ with $i \neq j$. We order these six matrices as follows:



We think of these six elements as arranged cyclically, because the relations among neighboring and next-neighboring elements in this cycle allow us to use the nilpotent subgroup criterion.

We then put $z_0 = z_6 = I + e_{12}$, $z_1 = I + e_{13}$, $z_2 = I + e_{23}$, $z_3 = I + e_{21}$, $z_4 = I + e_{31}$, $z_5 = I + e_{32}$ and $z_{i+6} = z_i$ for each i . Direct computations give the following three properties:

- (1) z_i commutes with z_{i+1} and z_{i-1} .
- (2) The commutator $[z_{i-1}, z_{i+1}]$ equals z_i^{-1} or z_i according as i is even or odd. For example, $[I + e_{12}, I + e_{23}] = I + e_{13}$. In particular, $\text{SL}_3(\mathbb{Z})$ is generated by $\{z_1, z_3, z_5\}$.
- (3) The elements z_{i-1} and z_{i+1} generate a nilpotent group N_i , and z_i belongs to the commutator subgroup of N_i .

Now suppose that $\text{SL}_3(\mathbb{Z})$ acts on a tree. Since z_i is in the commutator subgroup of N_i , z_i has a fixed point by Corollary 2.20. Since this is true for each $i \in \mathbb{Z}/6\mathbb{Z}$, we see that N_i is generated by elements which have fixed points, and it therefore has a fixed point by Corollary 2.19. In particular, $z_{i-1}z_{i+1}$ has a fixed point. Since z_1, z_3, z_5 and z_1z_3, z_3z_5, z_5z_1 all have fixed points, $\text{SL}_3(\mathbb{Z})$ must have a fixed point by Corollary 2.17. $\square$

In view of Theorem 2.12 this theorem implies:

Corollary 2.22. *The group $\text{SL}_3(\mathbb{Z})$ is not an amalgam.*

2.6. Mapping class groups and braid groups. In this section, we apply the techniques above to mapping class groups and braid groups, and prove similar results for them. We first introduce the generating set of mapping class groups and braid groups.

Let Σ_g be a closed orientable surface of genus g . The mapping class group $\text{Mod}(\Sigma_g)$ has the well known Lickorish generators: $3g - 1$ Dehn twists about curves shown in Figure 2. Note that every two simple closed curves in Figure 2 intersect at most once. We also recall that if α and β are isotopy classes of simple closed curves with $i(\alpha, \beta) = 1$, then the Dehn twists T_α and T_β satisfy $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$; if $i(\alpha, \beta) = 0$, then T_α and T_β commute with each other. For more relations in mapping class groups, see [FM12].

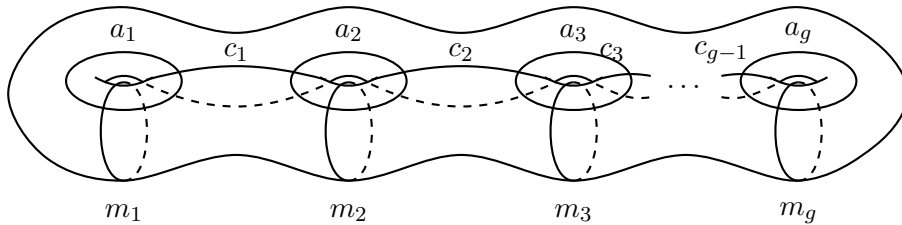


FIGURE 2. The Lickorish generating set for $\text{Mod}(\Sigma_g)$.

Also recall that the group B_m has the well known presentation

$$B_m = \langle \sigma_1, \dots, \sigma_{m-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, [\sigma_i, \sigma_j] = 1 \text{ if } |i - j| > 1 \rangle,$$

where σ_i is shown in Figure 3. See [FM12] for more details.

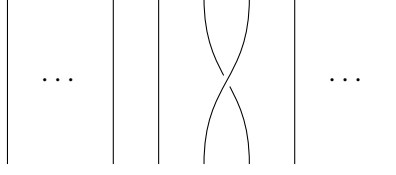


FIGURE 3. The generator σ_i for the braid group: the $(i+1)$ st strand passes in front of the i th strand.

Note that in both groups above, there are relations of the form $xyx = yxy$, which is called *braid relation*. In both mapping class groups and braid groups, every two generators either commute or braid with each other, so the fixed-point problem reduces to understanding commuting pairs and braiding pairs of tree automorphisms. Recall in Corollary 2.19 that when two group elements commute and both have fixed points, the group they generate also has a fixed point. In the following lemma, we prove a similar result when two elements braid with each other.

Lemma 2.23. *Let x, y be automorphisms of a tree X . If x, y satisfy $xyx = yxy$, and they both have fixed points, then x and y have a common fixed point. In particular, xy has a fixed point.*

Proof. Let A, B be the fixed point sets of x, y respectively. We consider the three subtrees A, yA , and $xyA = B$. It suffices to prove that $A \cap B \neq \emptyset$. If $A \cap B = \emptyset$, there is a unique geodesic of minimal length connecting A and B , denoted $[P, Q]$, where $P \in A, Q \in B$. Let $n \geq 1$ be the length of $[P, Q]$.

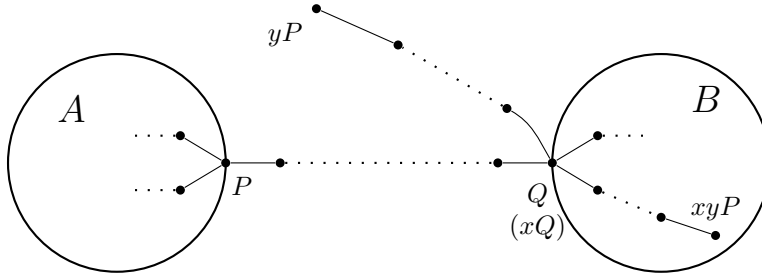
Note that

$$xyP = xyxP = yxyP = y(xyP),$$

so $xyP \in B$. By the minimal assumption on n , the geodesic $[yP, Q]$ must be disjoint with $[P, Q]$ except for the vertex Q . Then

$$d(xyP, P) = d(xyP, xP) = d(yP, P) = d(yP, Q) + d(P, Q) = 2n = d(P, xQ) + d(xQ, xyP).$$

Then $[P, xQ]$ and $[xQ, xyP]$ must be disjoint except for the vertex xQ . Since X is a tree, and $xyP \in B$, we must have $[P, Q] \subset [P, xyP]$. Note $d(P, xQ) = d(P, Q)$, so $xQ = Q$, contradiction.



□

Using this lemma, we can now prove property FA for mapping class groups and braid groups.

Theorem 2.24. *Let $g \geq 2$, and Σ_g be a closed orientable surface of genus g . Then $\text{Mod}(\Sigma_g)$ has property FA.*

Proof. Assume $\text{Mod}(\Sigma_g)$ acts on a tree X . Since all Lickorish generators are Dehn twists about nonseparating simple closed curves, they are all conjugate. Since fixed point behavior is invariant under conjugation, they must either all have a fixed point or none of them does.

In the first case, every two Lickorish generator x, y either commute with each other or satisfy the braid relation. If they commute with each other, by Corollary 2.19, xy has a fixed point. If they satisfy the braid relation, by Lemma 2.23, xy also has a fixed point. Then by Corollary 2.17, $\text{Mod}(\Sigma_g)$ must have a fixed point.

In the second case, each generator has a stable path (called *axis*) by Proposition 2.15. If two generators commute with each other, their axes must coincide. Since the commuting graph of the Lickorish generating set is connected as can be seen from Figure 2, all axes of the generators must coincide. Moreover, $\text{Mod}(\Sigma_g)$ acts on this axis by translations, giving a nontrivial homomorphism $\text{Mod}(\Sigma_g) \rightarrow \mathbb{Z}$. But $\text{Mod}(\Sigma_g)$ is a perfect group when $g \geq 3$, and any homomorphism of the group $\text{Mod}(\Sigma_2) \rightarrow \mathbb{Z}$ must factor through $\mathbb{Z}/10\mathbb{Z}$ [FM12], so such a homomorphism cannot exist, a contradiction. $\square$

We now switch focus to braid groups. Since the abelianization of the braid group on m strings B_m is isomorphic to $\mathbb{Z}$, it follows from Theorem 2.12 that B_m does not have property FA. However, if we consider the group $B_m/Z(B_m)$, we can bypass this obstacle and have property FA.

Theorem 2.25. *Let $m \geq 5$, and B_m be the braid group on m strings. Let $Z(B_m)$ denote the center of B_m . Then $B_m/Z(B_m)$ has property FA.*

Proof. We use the generating set given previously. The proof is almost the same as the proof of Theorem 2.24. The only difference is that when we obtain a nontrivial homomorphism $B_m/Z(B_m) \rightarrow \mathbb{Z}$, we still have a contradiction because the abelianization of $B_m/Z(B_m)$ is finite. $\square$

Remark 2.26. Note that the proof fails for B_m because there is no contradiction when all generators act as hyperbolic isometries. Also note that the proof fails again when $m = 3$ and $m = 4$ because when all generators act as hyperbolic isometries, the commuting relations are not enough to obtain that all axes of the generators coincide.

Remark 2.27. Moreover, $B_m/Z(B_m)$ is naturally related to the group $\text{Mod}(S_{0,m+1})$, the mapping class group of a sphere with $m + 1$ punctures. In fact, we have the exact sequence

$$1 \rightarrow Z(B_m) \rightarrow B_m \rightarrow \text{Mod}(\Sigma_{0,m+1}, \{p\}) \rightarrow 1,$$

where p is the puncture obtained from capping, and $\text{Mod}(S_{0,m+1}, \{p\})$ is subgroup of $\text{Mod}(S_{0,m+1})$ consisting of elements that fixes the puncture p . Hence we have

$$B_m/Z(B_m) \cong \text{Mod}(\Sigma_{0,m+1}, \{p\}).$$

3. GROUPS ACTING ON CAT(0) SPACES

In the previous section, we only took a tree as a combinatorial object. However, when equipped with the standard path metric, a tree serves as a metric space of negative curvature, because every geodesic triangle is degenerated to a tripod. In this section, we extend the results in section 2 to more general metric spaces of non-positive curvature, CAT(0) spaces. Indeed, CAT(0) spaces are geodesic metric spaces whose geodesic triangles are at least as thin as Euclidean geodesic triangles. Many properties of trees, such as the convexity of the metric, carry over to CAT(0) spaces. Thus we can give high dimensional analogues of the previous results on CAT(0) spaces.

The structure of this section parallels Section 2. We first define CAT(0) spaces and semisimple isometries in 3.1 and 3.2. Then we consider semisimple actions by abelian and nilpotent groups in 3.3, showing that they preserve flats. Next, we use convexity and Helly-type theorems in 3.5 to

turn fixed points of subgroups to global fixed points. We apply this directly to reflection groups and arithmetic groups in 3.6. Then we introduce Bridson's Bootstrap Lemma in 3.7 and Ample Duplication Criterion in 3.8. Finally, we apply these tools to mapping class groups and braid groups in 3.9.

3.1. CAT(0) space preliminaries. We first introduce the definition and some facts about CAT(0) spaces. See [BH99] for more background.

Definition 3.1. Let X be a geodesic metric space. A geodesic triangle Δ in X consists of three points $p, q, r \in X$ and three geodesics $[p, q], [q, r], [r, p]$. Let $\bar{\Delta} \subset \mathbb{E}^2$ be a triangle in the Euclidean plane with the same edge lengths as Δ and let $\bar{x} \mapsto x$ denote the map $\bar{\Delta} \rightarrow \Delta$ that sends each side of $\bar{\Delta}$ isometrically onto the corresponding side of Δ . One says that X is a CAT(0) space if for all Δ and all $\bar{x}, \bar{y} \in \bar{\Delta}$ the CAT(0) inequality $d_X(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$ holds; see Figure 4.

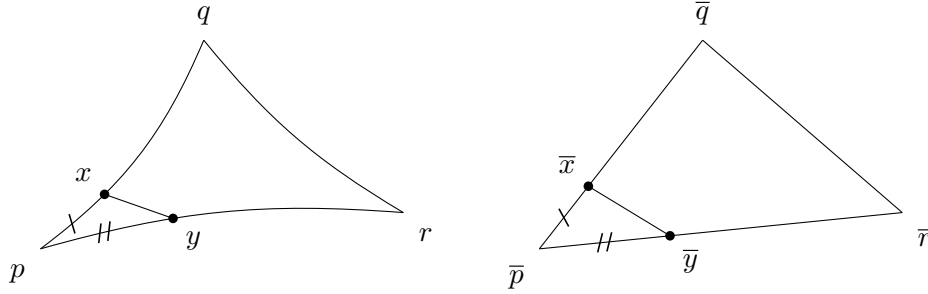


FIGURE 4. The CAT(0) inequality.

Note that in CAT(0) spaces there is a unique geodesic $[x, y]$ joining each pair of points $x, y \in X$. Thus CAT(0) spaces are always contractible.

We say that the metric on a metric space X is *convex*, if each pair of geodesics $c_1: [0, a_1] \rightarrow X$ and $c_2: [0, a_2] \rightarrow X$ with $c_1(0) = c_2(0)$ satisfies the inequality $d(c_1(ta_1), c_2(ta_2)) \leq t d(c_1(a_1), c_2(a_2))$ for all $t \in [0, 1]$. It is direct by computation that the metric on CAT(0) spaces is convex.

By the two properties above, we can see that the fixed-point set for an isometry of a CAT(0) space, if non-empty, is convex, hence is a CAT(0) space.

Example 3.2. To understand CAT(0) spaces, we highlight three classic examples:

- (1) Let T be a tree equipped with the standard path metric, where each edge is isometric to the unit interval $[0, 1]$, then T is a CAT(0) space.
- (2) The hyperbolic upper half space $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ equipped with the Riemannian metric $ds^2 = \frac{dx_1^2 + \dots + dx_n^2}{x_n^2}$ is a complete, simply connected n -dimensional Riemannian manifold with constant sectional curvature $K = -1$. One can check that it is a CAT(0) space.
- (3) Consider a finite 2-dimensional cell complex X formed by taking $k \geq 3$ Euclidean unit squares $E_1, \dots, E_k = [0, 1] \times [0, 1]$ and gluing them along a single common edge $e = \{0\} \times [0, 1]$ via the identity map. Endowed with the intrinsic path metric, X is a CAT(0) space.

3.2. Isometries of CAT(0) spaces. To understand the group actions on CAT(0) spaces, we first understand the isometries of CAT(0) spaces, similar to what we did in the case of trees.

Definition 3.3. Let X be a metric space and let γ be an isometry of X . The *displacement function* of γ is the function $d_\gamma : X \rightarrow \mathbb{R}_+$ defined by $d_\gamma(x) = d(\gamma x, x)$. The *translation length* of γ is the number $|\gamma| := \inf\{d_\gamma(x) \mid x \in X\}$. The set of points where d_γ attains this infimum, called *minimal set*, will be denoted $\text{Min}(\gamma)$.

When a group acts on trees, every automorphism either fixes a point or preserves an axis. In the former case, its minimal set is precisely its fixed point set, while in the latter, it is the axis. In CAT(0) spaces, however, a third possibility appears: the translation length may fail to be realized. Hence the isometries are divided into three categories based on whether it has a fixed point and whether its translation length can be realized.

Definition 3.4. Let X be a metric space. An isometry γ of X is called

- (1) *elliptic* if γ has a fixed point, that is $|\gamma| = 0$ and the infimum is attained,
- (2) *hyperbolic* if d_γ attains its infimum with $|\gamma| > 0$,
- (3) *parabolic* if d_γ does not attain its infimum, in other words if $\text{Min}(\gamma)$ is empty.

Since the fixed-point arguments below require actual minimal sets, we restrict attention to elliptic and hyperbolic actions, or, *semisimple* actions.

Definition 3.5. An isometry γ is called *semisimple* if it is elliptic or hyperbolic, in other words if $\text{Min}(\gamma)$ is non-empty.

Example 3.6. To understand group actions on CAT(0) spaces, we highlight three classic examples:

- (1) Let $n \geq 1$. The classic action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ is given by translations. More precisely, it is defined by $m \cdot x = x + m$ for every $m \in \mathbb{Z}^n$ and every $x \in \mathbb{R}^n$. For every nontrivial element $m \in \mathbb{Z}^n$, it acts as a hyperbolic isometry and its minimal set $\text{Min}(m) = \mathbb{R}^n$.
- (2) Let M be a n -dimensional closed hyperbolic manifold. Then its fundamental group $\pi_1(M)$ acts naturally on its universal cover $\mathbb{H}^n$ by deck transformations. For every nontrivial element $\gamma \in \pi_1(M)$, γ acts by a hyperbolic isometry, because M is compact. Its minimal set is precisely the unique geodesic axis of γ .
- (3) The group $\text{SL}_n(\mathbb{Z})$ acts naturally on the symmetric space $\text{SL}_n(\mathbb{R})/\text{SO}(n)$ by isometries. More precisely, $g \cdot (h \cdot \text{SO}(n)) = (gh) \cdot \text{SO}(n)$ for every $g \in \text{SL}_n(\mathbb{Z})$ and $h \in \text{SO}(n)$. This action is not semisimple because unipotent matrices act parabolically.

Now we can define property FA_n , the higher-dimensional analogue of property FA.

Definition 3.7. (Property FA_n , [Far09]). Let $n \geq 1$. A group Γ is said to have property FA_n if any isometric Γ -action on any n -dimensional, CAT(0) cell complex X has a global fixed point.

Remark 3.8. The property FA_1 corresponds with Serre's property FA.

While property FA_n is restricted to cell complexes, many natural CAT(0) spaces do not admit a cell complex structure. This motivates the definition of strong FA_n , which replaces cellular dimension with a homological condition on general complete CAT(0) spaces. More crucially, this also allows us to leverage powerful homological tools to analyze group actions.

Definition 3.9. (Strong FA_n , [Far09]). A group Γ is said to satisfy strong FA_n if any semisimple Γ -action on a complete CAT(0) space X satisfying *n-dimensionality* has a global fixed point, where *n-dimensionality* means $\tilde{H}_n(Y) = 0$ for all open subsets $Y \subseteq X$.

Remark 3.10. Since any isometric action on a cell complex as above must be semisimple [BH99], strong FA_n implies FA_n [Far09].

We first introduce some elementary facts about property FA_n and strong property FA_n , similar to that of property FA . See [Far09] for more details.

Proposition 3.11. *Let G be a group.*

- (1) *If G has property (strong) FA_n , then G has (strong) FA_m for all $m \leq n$.*
- (2) *If G has (strong) FA_n , then so does every quotient of G .*
- (3) *Let H be a normal subgroup of G . If both H and G/H have (strong) FA_n , then so does G .*
- (4) *If G contains a finite-index subgroup H that has (strong) FA_n , then so does G .*

In what follows we always assume that the space X is a metrically complete, n -dimensional, $\text{CAT}(0)$ space, and all actions on X are isometric and semisimple. So X satisfies the properties as in the definition of strong property FA_n .

3.3. Nilpotent groups. In this subsection, we develop a $\text{CAT}(0)$ analogue of Theorem 2.18. To understand group actions on $\text{CAT}(0)$, we first study the case of abelian groups, and then we extend the results to nilpotent groups. The following result is the $\text{CAT}(0)$ version of Theorem 2.15.

Proposition 3.12 (Abelian groups, Proposition 2.2 in [Far09]). *Let Γ be a finitely generated abelian group acting by semisimple isometries on a $\text{CAT}(0)$ space X . Then*

- (1) *$\text{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Min}(\gamma)$ is nonempty and splits as a product $Y \times \mathbb{R}^n$, $n \geq 0$ with Y nonempty and convex.*
- (2) *Γ leaves $\text{Min}(\Gamma)$ invariant, preserves the product structure, acts trivially on the first factor, and acts cocompactly by translations on the $\mathbb{R}^n$ factor. Moreover, $n \leq \text{rank}_{\mathbb{Q}} \Gamma$.*
- (3) *Any $g \in \text{Isom}(X)$ which normalizes Γ leaves $\text{Min}(\Gamma)$ invariant and preserves the isometric splitting $Y \times \mathbb{R}^n$. If g centralizes Γ , then the induced action on the $\mathbb{R}^n$ factor is by translations.*

The idea of proof is to first prove the result on finitely generated free abelian groups by induction on the rank of G , then generalize the result to finitely generated abelian groups by considering the G/E -action on $\text{Min}(E)$, where E is the subgroup of elliptic isometries in G . For a more detailed proof, see [BH99], [Far09].

Example 3.13. Let $\mathbb{Z}^2$ act on $\mathbb{R}^3$ by translations. More precisely, the action is defined by

$$(a, b) \cdot (y, x_1, x_2) = (y, x_1 + a, x_2 + b),$$

for every $(a, b) \in \mathbb{Z}^2$ and every $(y, x_1, x_2) \in \mathbb{R}^3$. Every non-trivial element in $\mathbb{Z}^2$ acts by a hyperbolic isometry, and $\text{Min}((a, b)) = \mathbb{R}^3$ for every $(a, b) \in \mathbb{Z}^2$. Hence $\text{Min}(\mathbb{Z}^2) = \bigcap_{(a, b) \in \mathbb{Z}^2} \text{Min}((a, b)) = \mathbb{R}^3$ splits as $\mathbb{R} \times \mathbb{R}^2$. Further, $\mathbb{Z}^2$ leaves $\text{Min}(\mathbb{Z}^2) = \mathbb{R}^3$ invariant, preserves the product structure, acts trivially on the first factor, and acts cocompactly by translations on the $\mathbb{R}^2$ factor.

The following result on nilpotent groups is the $\text{CAT}(0)$ analogue of Theorem 2.18.

Proposition 3.14 (Nilpotent groups, Proposition 2.3 in [Far09]). *Let N be a finitely-generated nilpotent group acting by semisimple isometries on a $\text{CAT}(0)$ space X . Then either*

- (1) *there is an N -invariant flat $\mathbb{R}^n$, $n > 0$ on which N acts by translations, or*
- (2) *$N \cdot x = x$ for some $x \in X$.*

Proof. We prove by induction on the length d of the lower central series for N . When $d = 1$ then N is abelian, and this is simply Proposition 3.12.

Now assume $d \geq 2$. Applying Proposition 3.12 to the center $Z(N) \neq N$ gives that $\text{Min}(Z(N)) = Y \times \mathbb{R}^n$, $n \geq 0$, where Y is a nonempty convex subset of X (hence is a complete $\text{CAT}(0)$ space). As N centralizes $Z(N)$, it leaves $\text{Min}(Z(N))$ invariant, preserves the product decomposition $Y \times \mathbb{R}^n$, and acts by translations on the $\mathbb{R}^n$ factor. In other words, the N -action on $\text{Min}(Z(N))$ is a product action

$\rho = \rho_1 \times \rho_2$, where $\rho_1 : N \rightarrow \text{Isom}(Y)$ and $\rho_2 : N \rightarrow \text{Isom}(\mathbb{R}^n)$. Note that we have $\rho_1(Z(N)) = \text{Id}$, so ρ_1 factors through an $N/Z(N)$ action on Y . Since $N/Z(N)$ is nilpotent of degree less than d , we can apply the inductive hypothesis. So $\rho_1(N/Z(N))$ leaves invariant some flat $\mathbb{R}^m$ in Y on which it acts by translations. Hence $\rho(N) = \rho_1(N) \times \rho_2(N)$ leaves invariant the flat $\mathbb{R}^m \times \mathbb{R}^n$ in $\text{Min}(Z(N))$, and acts on it via translations. $\square$

Similar to the case of nilpotent groups acting on trees (Corollary 2.17 and Corollary 2.20), we also have the following corollaries:

Corollary 3.15 (Corollary 2.4 in [Far09]). *Let a finitely generated nilpotent group $N = \langle s_1, \dots, s_n \rangle$ act by semisimple isometries on a CAT(0) space X . If each s_i has fixed points, then N has fixed point.*

Corollary 3.16 (Corollary 2.4 in [Far09]). *Let a finitely generated nilpotent group $N = \langle s_1, \dots, s_n \rangle$ act by semisimple isometries on a CAT(0) space X . Let N' be the commutator subgroup of N . If $g^m \in N'$ for some $m > 0$, then g has a fixed point.*

We now extend our results on nilpotent groups to more complicated groups by making use of the structure of their nilpotent subgroups.

3.4. Normalized and commuting subgroups. In this subsection, we explore when a group inherits fixed points from its subgroups. Recall that the fixed point set of a subgroup is a closed convex subset in the CAT(0) space, and global fixed point is given by the intersection of these convex subsets. We now start with the case of two subgroups with fixed points. The following lemma shows that the case of normalized subgroups is simple; see [Bri25] for a detailed proof.

Lemma 3.17 (Proposition 1.3 in [Bri25]). *Let Γ be a group, let $H_1, H_2 < \Gamma$ be subgroups, and suppose that H_2 normalizes H_1 . Whenever Γ acts by isometries on a complete CAT(0) space, if $\text{Fix}(H_1)$ and $\text{Fix}(H_2)$ are non-empty, then $\text{Fix}(H_1) \cap \text{Fix}(H_2)$ is non-empty.*

Proof. Since H_2 normalizes H_1 , the $\langle H_1, H_2 \rangle$ -orbit of any $x \in \text{Fix}(H_1)$ is simply the H_2 -orbit of x . This orbit is bounded because H_2 has fixed points. Hence $\langle H_1, H_2 \rangle$ has fixed points, that is, $\text{Fix}(H_1) \cap \text{Fix}(H_2)$ is non-empty. $\square$

We can apply the lemma to more subgroups and easily obtain the following:

Corollary 3.18. *Let X be a complete CAT(0) space. If the subgroups $H_1, \dots, H_\ell$ of $\text{Isom}(X)$ pairwise commute and $\text{Fix}(H_i)$ is non-empty for $i = 1, \dots, \ell$, then $\bigcap_{i=1}^\ell \text{Fix}(H_i)$ is non-empty.*

To record more complicated intersections among a collection of groups, we introduce a combinatorial object: the *nerve of a collection of sets*.

Definition 3.19 (Nerve of a collection of sets). Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a collection of sets. The *nerve* of $\mathcal{U}$, denoted by $\mathcal{N}(\mathcal{U})$, is the abstract simplicial complex whose vertex set is the index set A , where a finite non-empty subset $\sigma \subseteq A$ forms a simplex in $\mathcal{N}(\mathcal{U})$ if and only if the intersection of the corresponding sets is non-empty.

We know from Corollary 2.19 that when two elements commute and both have fixed points, the group they generated also has fixed point. It is then natural to extend this to a collection of commuting subgroups. We record the result in the language of nerves:

Lemma 3.20. *Let X be a complete CAT(0) space and let $S_1, \dots, S_\ell \subseteq \text{Isom}(X)$ be subsets such that $[s_i, s_j] = 1$ for all $s_i \in S_i, s_j \in S_j, i \neq j$. Let $\mathcal{N}_i$ be the nerve of the family $\mathcal{C}_i = (\text{Fix}(s_i) \mid s_i \in S_i)$, then the nerve $\mathcal{N}$ of $\mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_\ell$ is the join $\mathcal{N}_1 * \dots * \mathcal{N}_\ell$.*

3.5. Helly's theorem. Recall from 3.1 that fixed point sets of elliptic isometries are closed convex subsets of CAT(0) spaces. To prove a group has a global fixed point, we need to show that many such convex sets have common intersection. Then *Helly's theorem*, the mechanism that upgrades many partial intersections into one global intersection, naturally arises. We introduce some classic versions of Helly's theorem in this subsection.

Theorem 3.21 (Helly, 1913). *Let $\{X_i\}$ be any finite collection of nonempty, open convex sets in $\mathbb{R}^n$. If $X_{i_1} \cap \cdots \cap X_{i_{n+1}} \neq \emptyset$ for every $i_1 < \cdots < i_{n+1}$, then $\bigcap_i X_i \neq \emptyset$.*

However, we need a generalization of this theorem to CAT(0) spaces. Indeed, Debrunner gave the *Topological Helly Theorem* in [Deb70] by using methods in homology theory. Recall that a *homology cell* is a nonempty space X whose singular homology groups are isomorphic to those of a point.

Theorem 3.22 (Topological Helly Theorem). *Let X be a normal topological space with the property that every nonempty open subset $Y \subseteq X$ has $\tilde{H}_q(Y) = 0$ for all $q \geq n$. Let $\{X_i\}$ be any finite collection of nonempty closed homology cells in X . If the intersection of any r sets of this family is nonempty for $r \leq n + 1$, and is a homology cell for $r \leq n$, then $\bigcap_i X_i$ is a homology cell, in particular is nonempty.*

In particular, we will use the following version in this paper. We state the theorem in the language of nerve for sake of convenience; see [Bri25] for more details.

Theorem 3.23 (Metric Helly). *Let X be a convex metric space with $\dim(X) \leq d$, and let $\mathcal{C} = \{C_1, \dots, C_m\}$ be a collection of closed convex subspaces in X where $m \geq d + 2$. If the nerve $\mathcal{N}(\mathcal{C})$ contains the d -skeleton of the $(m - 1)$ -simplex Δ^{m-1} , then $\mathcal{N}(\mathcal{C}) = \Delta^{m-1}$.*

3.6. Reflection groups and arithmetic groups. In this subsection, we use Helly's theorem to prove property FA_n for reflection groups and arithmetic groups. In the case of reflection groups, we first find a collection of finite subgroups that generate the whole group, which must have fixed points, and then apply Helly's theorem to yield a global fixed point. In the case of arithmetic groups, we replace the finite subgroups with nilpotent subgroups and apply the same strategy.

Theorem 3.24 (Section 5.3 in [Far09]). *Let Γ be a Euclidean or hyperbolic reflection group about a n -simplex. Then Γ has property FA_{n-1} .*

Proof. Let r_i be the reflection about the i -th face, for $i = 1, \dots, n + 1$. Then $\Gamma = \langle r_1, \dots, r_{n+1} \rangle$. Suppose that Γ acts by semisimple isometries on a complete CAT(0) space X of dimension $n - 1$. Since Γ is discrete, the group $\langle r_1, \dots, \widehat{r_j}, \dots, r_{n+1} \rangle = \text{Stab}_\Gamma(v_j)$ is finite, where v_j is the unique vertex disjoint with the j -th face, for $j = 1, \dots, n + 1$. Recall that any finite group action on a CAT(0) space X has a global fixed point [BH99]. Then $\forall I \subset \{1, \dots, n + 1\}$ with $|I| = n$, we have $\bigcap_{i \in I} \text{Fix}(r_i) \neq \emptyset$. By Theorem 3.23, $\bigcap_{i=1}^{n+1} \text{Fix}(r_i) \neq \emptyset$, in other words Γ has a global fixed point. $\square$

Next, we prove in the case of $\text{SL}_n(\mathbb{Z})$ as an example of arithmetic groups. For more examples on arithmetic groups, S -arithmetic groups and Chevalley groups, see [Far09].

Theorem 3.25. *Let $n \geq 3$. Then $\text{SL}_n(\mathbb{Z})$ has property FA_{n-2} .*

Proof. Suppose that $\text{SL}_n(\mathbb{Z})$ acts by semisimple isometries on a complete CAT(0) space X of dimension $n - 2$. Let $g_i = 1 + e_{i,i+1}$, for $i = 1, \dots, n - 1$, and $g_n = 1 + e_{n,1}$, where e_{ij} is the defined in the proof of Theorem 2.21. One can check that the group $\langle g_1, \dots, \widehat{g_i}, \dots, g_n \rangle$ is nilpotent, and that g_i is in the commutator subgroup of $\langle g_1, \dots, \widehat{g_i}, \dots, g_n \rangle$, for $i = 1, \dots, n$. Then Corollary 3.16 gives that g_i has a fixed point, and Corollary 3.15 gives that $\langle g_1, \dots, \widehat{g_i}, \dots, g_n \rangle$ has a fixed point. Then $\forall I \subset \{1, \dots, n\}$ with $|I| = n - 1$, we have $\bigcap_{i \in I} \text{Fix}(g_i) \neq \emptyset$. By Theorem 3.23, $\bigcap_{i=1}^n \text{Fix}(g_i) \neq \emptyset$, in other words $\text{SL}_n(\mathbb{Z})$ has a global fixed point. $\square$

3.7. Bootstrap lemma. However, not all groups can be generated by a collection of finite groups or nilpotent groups. Thus we might have to add more assumptions on the group action to prove that there exist a global fixed point. Combining the results in Lemma 3.20 and 3.5, we obtain a more widely applicable fixed-point criterion: *Bootstrap lemma*. Roughly speaking, it states that on a low-dimensional CAT(0) space, if several mutually commuting families of isometries possess fixed points respectively, then at least one family must possess a common fixed point on every finite subset.

Lemma 3.26 (Bootstrap Lemma, Proposition 4.2 in [Bri25]). *Let $k_1, \dots, k_n$ be positive integers and let X be a complete CAT(0) space of dimension less than $k_1 + \dots + k_n$. Let $S_1, \dots, S_n \subseteq \text{Isom}(X)$ be subsets with $[s_i, s_j] = 1$ for all $s_i \in S_i, s_j \in S_j, i \neq j$. If the subgroup generated by each k_i -element subset of S_i has a fixed point in X , for $i = 1, \dots, n$, then for some i every finite subset of S_i has a common fixed point.*

Proof. The intuition is that if no S_i has a common fixed point for every finite subset, the nerves contain boundaries of simplices whose join gives a sphere too large to occur. Now we give the detailed proof below.

If the conclusion is false, then for $i = 1, \dots, n$ there would be a smallest integer $k'_i \geq k_i$ such that some $(k'_i + 1)$ -element subset $T_i = \{s_{i,1}, \dots, s_{i,k'_i+1}\}$ in S_i did not have a fixed point. Since any k'_i elements of T_i have a common fixed point, the nerve of the family $\mathcal{C}_i = (\text{Fix}(s_{i,j}) \mid j = 1, \dots, k'_i + 1)$ would be the boundary of a k'_i -simplex $\partial\Delta_{k'_i}$. Hence, by Lemma 3.20, the nerve of $\mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$ would be the join $\partial\Delta_{k'_1} * \dots * \partial\Delta_{k'_n}$, which is homeomorphic to a sphere of dimension $(\sum_{i=1}^n k'_i) - 1$. Since $(\sum_{i=1}^n k'_i) - 1 \geq \dim X$, the $(\dim X)$ -skeleton is contained in the nerve $\mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$. But by Theorem 3.23, the nerve cannot be a sphere. Contradiction. $\square$

To overcome that the conclusion above only applies to some S_i , we make all S_i conjugate to each other and obtain the following:

Corollary 3.27 (Conjugate Bootstrap, Corollary 4.6 in [Bri25]). *Let k and n be positive integers and let X be a complete CAT(0) space of dimension less than nk . Let $S_1, \dots, S_n$ be conjugates of a subset $S \subseteq \text{Isom}(X)$ with $[s_i, s_j] = 1$ for all $s_i \in S_i, s_j \in S_j, i \neq j$. If each k -element subset of S has a fixed point in X , then so does each finite subset of S and of $S_1 \cup \dots \cup S_n$.*

3.8. Ample Duplication Criterion. In what follows we give a general scheme on fixed-point properties, called *ample duplication*, by combining Lemma 3.17 and Corollary 3.27. Although it is difficult to prove the existence of a global fixed point directly for a large generating set, we have already proved the existence of fixed points for groups with simple structure, such as abelian groups. Intuitively, ample duplication criterion proves the existence of global fixed points given that the group has a large number of subgroups with fixed points. We first introduce a technical definition.

Definition 3.28 (Definition 5.1 in [Bri25]). Given a group Γ and positive integers d and k_0 , we say that a finite generating set $\mathcal{A}$ for a subgroup $\Lambda < \Gamma$ has *ample duplication for dimension d , with base k_0* , if there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that the following conditions both hold:

- (1) Each subset $S \subseteq \mathcal{A}$ of cardinality $|S| > k_0$ can either be written as a disjoint union $S = S_1 \sqcup S_2$ where the S_i are non-empty and $\langle S_1 \rangle$ normalizes $\langle S_2 \rangle$, or else there are at least $f(|S|)$ commuting conjugates of $\langle S \rangle$ in Γ ;
- (2) $d < (k - 1)f(k)$ for $k = k_0 + 1, \dots, \min\{d + 1, |\mathcal{A}|\}$.

When these conditions hold, $f(n)$ is said to be an *ample duplication function*.

Remark 3.29. Condition (1) above enables global fixed points to arise from fixed points of the subgroups. While the former is exactly the condition in Lemma 3.17, the latter is the condition in Corollary 3.27. Condition (2) comes from the dimension restriction in Corollary 3.27.

Theorem 3.30 (Ample Duplication Criterion, Theorem 5.2 in [Bri25]). *Let Γ be a group acting by isometries on a complete CAT(0) space X of dimension at most d , and let $\Lambda < \Gamma$ be a subgroup with a finite generating set $\mathcal{A}$ that has ample duplication for dimension d , with base k_0 . If $\langle S \rangle$ has a fixed point for every $S \subseteq \mathcal{A}$ with $|S| \leq k_0$, then Λ has a fixed point in X .*

Proof. We argue by induction on $|S|$ to show that $\langle S \rangle$ has a fixed point for every $S \subseteq \mathcal{A}$. This is already true for $|S| \leq k_0$. Suppose now that $|S| = k > k_0$ and that smaller subsets of $\mathcal{A}$ all have common fixed points.

If $S = S_1 \sqcup S_2$, as in condition (1) in Definition 3.28, then the common fixed points of $\langle S_1 \rangle$ and $\langle S_2 \rangle$ provided by Lemma 3.17 are fixed points for $\langle S \rangle$. If not, then condition (1) provides $l := f(k)$ commuting conjugates of $\langle S \rangle$, say $\Sigma_1, \dots, \Sigma_l$. By Conjugate Bootstrap (Corollary 3.27), there is a fixed point for $\langle S \rangle$ when $(k-1)f(k) > d$ holds, which it does by condition (2), unless $k \geq d+2$. In this case, we can apply Theorem 3.23. $\square$

3.9. Mapping class groups and braid groups. We can now apply the techniques established above to prove the fixed point properties first for mapping class groups and then for braid groups.

Theorem 3.31. *Let $g \geq 1$. $\text{Mod}(\Sigma_g)$ has property FA_n when $n < g$.*

Before we apply the Ample Duplication Criterion, we need the following propositions to prove that the assumption in Theorem 3.30 is true for mapping class groups. More precisely, we will prove that every generator in $\text{Mod}(\Sigma_g)$ has a fixed point.

Proposition 3.32 (Theorem B in [Bri10]). *Let Σ be an orientable surface of finite type with genus $g \geq 3$. If $\text{Mod}(\Sigma)$ acts by semisimple isometries on a complete CAT(0) space, any Dehn twist about a nonseparating simple closed curve in $\text{Mod}(\Sigma)$ acts as an elliptic isometry.*

Proof. Suppose that $\text{Mod}(\Sigma)$ acts by semisimple isometries on a complete CAT(0) space X , and T is the Dehn twist about a nonseparating simple closed curve c . If T acts as a hyperbolic element, then the union of axes of T splits as $Y \times \mathbb{R}$, and the centralizer $C_{\text{Mod}(\Sigma)}(T)$ preserves this splitting and acts by translations on the second factor [BH99]. Hence we have a nontrivial homomorphism $C_{\text{Mod}(\Sigma)}(T) \rightarrow \text{Isom}_+(\mathbb{R})$. Since $\text{Isom}_+(\mathbb{R})$ is an abelian group, T must have infinite order in the abelianization of $C_{\text{Mod}(\Sigma)}(T)$.

The centralizer of T in $\text{Mod}(\Sigma)$ consists of mapping classes of homeomorphisms that leave c invariant, which is a homomorphic image of the mapping class group of the surface obtained by cutting Σ along c . Since Σ has genus at least 3, the cut surface has genus at least 2. But the mapping class group of such a surface has finite abelianization [FM12], then so does the centralizer of T in $\text{Mod}(\Sigma)$, which contradicts with the last paragraph. $\square$

Now we need a large number of conjugate subsets of the generators, so that we can apply the ample duplication criterion. Let Lick denote the set of loops shown in Figure 2. We say that a subset $S \subset \text{Lick}$ is *connected* if the union $U(S)$ of the loops in S is connected. Recall that when two subsets of Lick are not connected to each other, the two subgroups generated by the corresponding Dehn twists commute. In particular, they normalize each other. Hence it suffices to prove the existence of conjugate subsets on a subgroup generated by Dehn twists corresponding to a connected subset of Lick . By analyzing the Lickorish generators, one can prove the following fact by induction; see [Bri12] for a detailed proof.

Proposition 3.33. *Let $S \subset \text{Lick}$ be a connected subset.*

- (1) *If $|S| = 2\ell$ is even, then $U(S)$ is either contained in a sub-surface of genus ℓ with 1 boundary component, or else in a non-separating sub-surface of genus at most $\ell - 1$ with 3 boundary components.*

- (2) If $|S| = 2\ell + 1$ is odd, then $U(S)$ is either contained in a non-separating subsurface of genus ℓ with at most 2 boundary components, or else in a non-separating sub-surface of genus at most $(\ell - 1)$ that has at most 3 boundary components.

With all the preparation above, we can now apply the ample duplication criterion directly to prove Theorem 3.31.

Proof of Theorem 3.31. The case $g = 1$ is trivial. The case $g = 2$ is already proved in Theorem 2.24. Now assume $g \geq 3$. By Proposition 3.33, one can check that the Lickorish generators of $\text{Mod}(\Sigma_g)$ have ample duplication for dimension $g - 1$, with base 1 and duplication function

$$f(k) = \begin{cases} \lfloor 2g/k \rfloor & k \text{ even,} \\ \lfloor 2(g-1)/(k-1) \rfloor & k \text{ odd.} \end{cases}$$

By Proposition 3.32, $\langle T \rangle$ has a fixed point for every $T \in \text{Lick}$. Hence we are done by Ample Duplication Criterion (Theorem 3.30). $\square$

Now we switch focus from mapping class groups to braid groups. However, we cannot prove property FA_n for braid groups directly, because the generators in braid groups do not necessarily have a fixed point. In other words, Theorem 3.32 fails in the case of braid groups, because the abelianization of the centralizer of a generator is not finite. Hence we can only prove property FA_n under certain assumptions:

Theorem 3.34 (Corollary 5.7 in [Bri25]). *Suppose that B_m acts by isometries on a complete CAT(0) space X .*

- (1) *If each generator σ_i has a fixed point and $\dim X < \lfloor m/3 \rfloor$, then B_m has a fixed point.*
(2) *If one of the subgroups $\langle \sigma_i, \sigma_{i+1} \rangle \cong B_3$ has a fixed point and $\dim X < 2\lfloor m/4 \rfloor - \delta_m$, then B_m has a fixed point, where $\delta_m = 0$ if $m \equiv 2$ or $3 \pmod{4}$, and $\delta_m = 1$ if $m \equiv 0$ or $1 \pmod{4}$.*

Recall that the braid group on m strings B_m has the well-known presentation

$$B_m = \langle \sigma_1, \dots, \sigma_{m-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, [\sigma_i, \sigma_j] = 1 \text{ if } |i - j| > 1 \rangle,$$

see [FM12]. Let $\sigma_m = \sigma_1 \cdots \sigma_{m-2} \sigma_{m-1} \sigma_{m-2}^{-1} \cdots \sigma_1^{-1}$, then we have the relation $[\sigma_i, \sigma_j] = 1$ with indices taken cyclically, if i and j are not adjacent in the cyclic order. In what follows we consider the generating set $\underline{\sigma} = \{\sigma_1, \dots, \sigma_m\}$ for sake of symmetry. With this generating set, we can prove Theorem 3.34 using the same strategy in the proof of Theorem 3.31.

Proof of Theorem 3.34. It suffices to prove that the generating set $\underline{\sigma} = \{\sigma_1, \dots, \sigma_m\}$ for B_m has ample duplication for dimension $\lfloor m/3 \rfloor - 1$ with base 1 and duplication function

$$f_m(k) = \lfloor m/(k+1) \rfloor,$$

has ample duplication for dimension $2\lfloor m/4 \rfloor - 2$ with base 2 and the same duplication function, and has ample duplication for dimension $2\lfloor m/4 \rfloor - 1$ with base 2 if $m \equiv 2$ or $3 \pmod{4}$.

Given a proper subset $S_I = \{\sigma_i \mid i \in I\}$ of $\underline{\sigma}$ with $|I| = k \geq 2$, we may conjugate by a power of $\sigma_1 \cdots \sigma_{m-1}$ to assume that $\sigma_m \notin I$. Then, either I can be written as a disjoint union $I_1 \sqcup I_2$ with $\max I_1 < \min I_2 - 1$, or else I is an interval $[i_0, i_0 + k - 1] \cap \mathbb{N}$. In the first case, $\langle S_{I_1} \rangle$ commutes with $\langle S_{I_2} \rangle$, and in the second case we have $f_m(k) := \lfloor m/(k+1) \rfloor$ commuting conjugates of $\langle S_I \rangle$ in B_m , namely the subgroups generated by $J_0, \dots, J_{f(k)-1}$ where $J_r = \langle \sigma_{r(k+1)+1}, \dots, \sigma_{r(k+1)+k} \rangle$.

For the first assertion, we take $k_0 = 1$ in the Ample Duplication Criterion and check that $\lfloor m/3 \rfloor \leq (k-1)f_m(k)$ for $k = 2, \dots, \lfloor m/3 \rfloor$.

For the second assertion, we take $k_0 = 2$ and check that $2\lfloor m/4 \rfloor - 1 \leq (k-1)f_m(k)$ for $k = 3, \dots, 2\lfloor m/4 \rfloor - 1$.

For the third assertion, we can also take $k_0 = 2$ and check that $2\lfloor m/4 \rfloor \leq (k-1)f_m(k)$ for $k = 3, \dots, 2\lfloor m/4 \rfloor$. $\square$

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