

Chapter II.6 Isometries of CAT(0) Spaces.

Example: $\text{Isom}(\mathbb{F}^n) \cong \mathbb{R}^n \times O(n)$.

$$\text{Isom}(H^n) \cong O(n, 1)_+ := \text{subgroup of } O(n, 1) \text{ preserving } H^n.$$

Where $O(n, 1) = \{A \in M_{n \times n}(\mathbb{R}) : A^T J A = J, J = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}\}.$

Scheme =

Part 1 = Individual Isometries.

Part 2: $\text{Isom}(X)$ & Clifford translations.

6.1 Individual Isometries.

Categorize into 3 classes: elliptic, hyperbolic, parabolic using ...

Def 6.1: X metric space, γ isometry of X .

displacement function of V is $d_V: X \rightarrow \mathbb{R}_+$, $d_V(x) = d(x, V(x))$.

translation length of γ is $|\gamma| := \inf \{ d_Y(x) : x \in X \}$.

$$M_m(Y) := \{x \in X : d_Y(x) = |Y|\}.$$
$$\text{Min}(T) := \bigcap_{Y \in T} \text{Min}(Y).$$

V is called semi-simple $\nexists$ $\text{Min}(V) \neq \phi$.

P — — — If $\forall \epsilon \in T$, r semi-simple.

Prop 6.2: X metric space, f isometry of X , P group of isometries.

(1) $\text{Min}(Y) \ni Y$ -invariant, $\text{Min}(\bar{Y}) \ni \bar{Y}$ -invariant.

(2) α geometry of X , $|r| = |\alpha r \alpha^{-1}|$, $\text{Min}(\alpha r \alpha^{-1}) = \alpha(\text{Min}(r))$.

$N \triangleleft T$, $\text{Min}(N)$ is T -invariant.

(3) $X \text{ CAT}(0)$, d_V is convex, hence $\text{Min}(V)$ is a closed convex set.

(4) $C \subset X$ non-empty, complete, convex, V -invariant.

then $|V| = |V|_C$, V semi-simple $\Leftrightarrow V|_C$ semi-simple.

thus $\text{Min}(V) \neq \emptyset \Leftrightarrow C \cap \text{Min}(V) \neq \emptyset$.

(1) is trivial, (3) = convexity of distance function in $\text{CAT}(0)$ spaces

(4) p : projection onto C , Prop 2.4 $\Rightarrow p(Vx) = V(p(x))$.

$$\Rightarrow d_C(p(x), p(y)) = d(p(x), p(y)) \leq d(Vx, Vy)$$

$$\Rightarrow p(\text{Min}(V)) = \text{Min}(V) \cap C = \text{Min}(V|_C).$$

□

Def 6.3 : X metric space, V isometry of X .

(1) V is elliptic, $\forall V$ has a fixed point.

(2) V is hyperbolic, $\forall |V| > 0$.

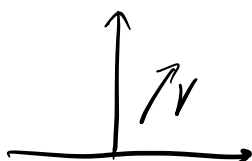
(3) V is parabolic, $\forall \text{Min}(V) = \emptyset$.

} semi-simple.

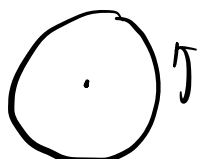
Example: H^n , $\{x \in \mathbb{R}^n : x_n > 0\}$,

hyperbolic: $V : x \mapsto \lambda x$, $\lambda \neq 1$, $\lambda > 0$, $|V| = |\log \lambda|$, $\text{Min}(V) = \{x : x_i = 0, i \leq n-1\}$.

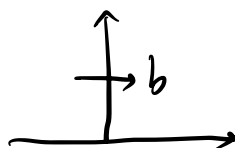
$\{\alpha g \alpha^{-1} : \alpha \in \text{Isom}(H^n), g \text{ orthogonal transformation that fixes the axis of } V \text{ pointwise}\}$.



hyperbolic



elliptic



parabolic

• Semi-simple Isometries

Prop 6.7: X complete CAT(0), γ isometry of X .

(elliptic) γ elliptic $\Leftrightarrow \gamma$ has a bounded orbit.

And $\forall \gamma^n$ elliptic, $n \neq 0$, then γ elliptic.

pf: $\Leftarrow$ Prop 2.7 (unique center) = $\text{Orb}(x)$ bounded, center of $\text{Orb}(x)$ is fixed
 γ^n fixes $x \Rightarrow \text{Orb}_\gamma(x)$ finite. $\square$

(hyperbolic)

Thm 6.8: X CAT(0). γ isometry of X .

(1) γ hyperbolic $\Leftrightarrow \exists$ geodesic line $C: \mathbb{R} \rightarrow X$, s.t. $\gamma(C(t)) = C(t+a)$, $a > 0$.

$C(\mathbb{R})$ is called an axis of γ , $a = |\gamma|$.

(2) X complete, γ^m hyperbolic, $m \neq 0$, then γ hyperbolic.

Now assume γ hyperbolic.

(3) the axes are parallel to each other, their union is $\text{Min}(\gamma)$.

(4) $\text{Min}(\gamma) \stackrel{\text{iso}}{\cong} Y \times \mathbb{R}$, $\gamma|_{\text{Min}(\gamma)}: (y, t) \mapsto (y, t + |\gamma|)$.

(5) α isometry, commutes with γ .

Then $\text{Min}(\gamma)$ is α -invariant, $\alpha|_{Y \times \mathbb{R}} = (\alpha', \alpha'')$,

where α' is an isometry of Y , α'' a translation of $\mathbb{R}$.

pf: (1) $\Leftarrow$ Prop 6.2(4).

$\Rightarrow$: $\forall x \in \text{Min}(\gamma)$, x lies in axis $\bigcup_{n \in \mathbb{Z}} [\gamma^n(x), \gamma^{n+1}(x)]$

it suffices to show $[x, \gamma(x)]$ is the concatenation of $[x, \gamma(x)]$, $[\gamma(x), \gamma^2(x)]$

$x, \gamma(x) \in \text{Min}(\gamma)$, $\text{Min}(\gamma)$ convex, $m = \text{midpoint of } [x, \gamma(x)]$.

$\Rightarrow m \in \text{Min}(\gamma) \Rightarrow \text{dim}_\gamma(m) = \text{dim}_\gamma(x) + \text{dim}_\gamma(\gamma(x), \gamma(m))$.

(3) $C, C': \mathbb{R} \rightarrow X$ axes of V . $V(C(t)) = C(t+|V|)$.

$\Rightarrow f(t) := d(C(t), C'(t))$ is periodic of period $|V|$.

$f(t)$ convex, bounded $\Rightarrow$ constant.

(4) Apple Thm 2.14: $\text{Min}(V) = Y \times \mathbb{R}$, $\{y\} \times \mathbb{R}$ is an axis

$\Rightarrow r|_{\text{Min}(V)} : (y, t) \mapsto (y, t+|V|)$

Thm 2.14: X CAT(0), $C: \mathbb{R} \rightarrow X$ geodesic

- (1) $X_C := \bigcup \{C'(\mathbb{R}) : C' \parallel C\} \subseteq X$ is convex.
- (2) $X_C^\circ = X_C \cap \pi^{-1}(C(\mathbb{R}))$ is convex, $\pi: X \rightarrow C$.

then $X_C \stackrel{\text{iso}}{\cong} X_C^\circ \times \mathbb{R}$

(5) Prop 6.2 (2) $\Rightarrow \text{Min}(V)$ is α -invariant.

commute $\Rightarrow \alpha$ takes an axis to an axis.

By Prop I.5.3: α splits as (α', α'') , $\alpha' \in \text{Isom}(Y)$, $\alpha'' \in \text{Isom}(\mathbb{R})$.

α'' commutes with translation defined by $V \Rightarrow \alpha''$ is translation.

(2) V^m hyperbolic $\Rightarrow \text{Min}(V^m) \stackrel{\text{iso}}{\cong} Y \times \mathbb{R}$.

$V|_{Y \times \mathbb{R}} = (V', V'')$, $(V')^m = \text{Id}_Y$, V'' is a non-trivial translation.

$\text{Min}(V^m)$ closed, convex $\Rightarrow Y$ closed, convex $\Rightarrow Y$ complete CAT(0)

Prop 6.7 $\Rightarrow V'$ elliptic, fixes x

$\Rightarrow r|_{\{x\} \times \mathbb{R}} : (x, t) \mapsto (x, t+a)$. □

6.2 General Structure of $\text{Isom}(X)$.

Prop 6.10: Fawcett's theorem: P acts properly by isometries on X .

Skip.

Thm 6.12: X CAT(0), Γ finitely generated group acting by isometries on X .

Γ has central subgroup $A \cong \mathbb{Z}^n$ acts faithfully by hyperbolic isometries.
then $\exists H \leq \Gamma$ with finite index, contains A as a direct factor.

pf: $\alpha \in A$ nontrivial, Prop 6.8 $\Rightarrow \text{Min}(\alpha) = Y \times \mathbb{R}$ is Γ -invariant.

$\forall \gamma \in \Gamma, \gamma = (\gamma', \gamma''), \gamma' \in \text{Isom}(Y), \gamma''$ is a translation of $\mathbb{R}$

$\psi: \Gamma \rightarrow \text{Isom}(\mathbb{R}^n), \gamma \mapsto \gamma'$

Γ finitely generated $\Rightarrow \psi(\Gamma) \cong \mathbb{Z}^m, m \in \mathbb{N}$.

$\Rightarrow \psi: \Gamma \rightarrow \mathbb{Z}^m$ surjective.

$\psi(\alpha)$ nontrivial $\Rightarrow \psi(A)$ nontrivial

compose ψ with projection of $\mathbb{Z}^m$ onto a suitable direct summand,

s.t. $\phi: \Gamma \rightarrow \mathbb{Z}, \phi(A)$ nontrivial.

Choose $a \in A$ s.t. $\phi(a)$ generates $\phi(A)$. $[\Gamma: H_0] = [\phi(\Gamma): \phi(A)]$

$H_0 = \phi^{-1}(\phi(A))$ is a subgroup with finite index.

$\Rightarrow H_0 = \ker \phi \times \langle a \rangle, \quad 0 \rightarrow \ker \phi \xrightarrow{\psi} H_0 \xrightarrow{\phi} \langle \phi(a) \rangle \rightarrow 0, \quad H_0 \cong \ker \psi \times \langle a \rangle,$

$\Rightarrow A = A' \times \langle a \rangle$, where $A' = A \cap \ker \phi$. $\langle a \rangle \cong \mathbb{Z}$ a : central.

Induction on rank of A : $\exists H' \leq \ker \phi$ with finite index, $H' = B \times A'$

then: let $H = H' \times \langle a \rangle = B \times A' \times \langle a \rangle = B \times A$. $\square$

6.3 Clifford Translations & Euclidean de Rham Factor.

Def 6.14: X metric space, $V \in \text{Isom}(X)$,

V is called a Clifford translation, if d_V constant, i.e. $\text{Min}(V) = X$.

Recall: simply connected complete Riemann mfd of non-positive curvature

admits a canonical splitting $M = \underbrace{N}_{\text{not contain Euclidean factor}} \times \underbrace{\mathbb{E}^n}_{\text{called Euclidean de Rham Factor}}$

Thm 6.15: X CAT(0),

(1) γ nontrivial Clifford translation of X .

then $X = Y \times \mathbb{R}$, $\gamma(y, t) = (y, t + |\gamma|)$, $\forall y \in Y$, $\forall t \in \mathbb{R}$

(2) $X = X' \times X''$, then Clifford translation γ preserves splitting.

$\gamma = (\gamma', \gamma'')$, γ', γ'' is Clifford translation of X', X'' respectively.

(3) $H = \langle \text{Clifford translation} \rangle \leq \text{Isom}(X)$, H abelian.

(4) H is a pre-Hilbert space (inner product space).

where the norm of γ is its translation length $|\gamma|$.

(5) X complete, H is Hilbert space.

(6) If space H complete, then X admits a splitting $Y \times H$.

s.t. $\forall y \in Y$, $\{y\} \times H$ is $H \cdot y$ (orbit of y under the group H).

Furthermore, $\forall \gamma \in \text{Isom}(X)$ preserves splitting.

Remark: H is the analogue for CAT(0) spaces

of Euclidean de Rham factor in Riemann geometry.

pf: (1) Prop 6.8 (4)

(2) γ nontrivial Clifford translation,

consider foliation of X by γ -axes.

If axes project to points, then $\gamma = (\text{id}, \gamma'')$, γ'' Clifford

Otherwise, projection of axes give a foliation of X' by parallel geodesic lines.

$$V(x_0', x_0'') = (x_1', x_1'').$$

V' translate parallel geodesics a distance $d(x_0', x_1')$, V'' similar.
then (V', V'') Clifford, and on axis through (x_0', x_0'') . $(V', V'') = V$.
(1) $\Rightarrow V = (V', V'')$.

(3) α, β nontrivial Clifford translations.

$$(1) \Rightarrow X = Y \times \mathbb{R}, \alpha = \text{Id}, \tau_\alpha$$

$$(2) \Rightarrow \beta = (\beta', \tau_\beta), \beta' \text{ Clifford}, \tau_\beta \text{ translation of } \mathbb{R}.$$

$$\text{Thus, } \alpha\beta = (\beta', \tau_\alpha \tau_\beta) = \beta\alpha$$

Rmk: If β' nontrivial, (1) $\Rightarrow Y = Y' \times \mathbb{R} \Rightarrow X = Y' \times \mathbb{E}^2$.

α, β projects to id on Y' , translation on $\mathbb{E}^2$.

(4) Vector space structure:

Scalar multiplication: V nontrivial Clifford. $\lambda \in \mathbb{R}$,

$$X = Y \times \mathbb{R}, V: (y, t) \mapsto (y, t + |V|)$$

$$\lambda V: (y, t) \mapsto (y, t + \lambda |V|).$$

easy to check it's vector space: Rmk above imply $\lambda(\alpha + \beta) = \lambda\alpha + \alpha\beta$

easy to check $|V|$ is a norm.

parallelogram law: $\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$, Implied by Rmk.

$$\text{Induce } \langle \alpha, \beta \rangle = \frac{1}{4}(\|\alpha + \beta\|^2 - \|\alpha - \beta\|^2)$$

$$(5) d(\alpha(x), \beta(x)) = |\alpha^{-1}\beta|, \forall x \in X$$

$\Rightarrow$ If $\{\alpha_n\}$ Cauchy in H , then $\{\alpha_n(x)\}$ Cauchy in X . $\forall x \in X$

$$\lim_{n \rightarrow \infty} \alpha_n(x) \triangleq \alpha(x), d(x, \alpha(x)) = \lim_{n \rightarrow \infty} d(x, \alpha_n(x)) = \lim_{n \rightarrow \infty} |\alpha_n|.$$

$\Rightarrow X \mapsto \alpha(x)$ is Clifford. and $|\alpha^{-1}\alpha_n| \xrightarrow{n \rightarrow \infty} 0$.

(6) $x_0 \in X$, p denote projection of X onto Hx_0 , $\gamma := p^{-1}(x_0)$.

$\gamma: Y \times H \rightarrow X$, $(y, \alpha) \mapsto \alpha(y)$.

surjective: $\forall x \in X$, $p(x) \in Hx_0 \Rightarrow p(x) = \alpha(x_0)$, $\alpha \in H$

$$y := \alpha^{-1}(x) \Rightarrow p(y) = \alpha^{-1}(p(x)) = x_0 \Rightarrow y \in \gamma$$

injective: $y_1, y_2 \in Y$, $\alpha_1, \alpha_2 \in H$, $\alpha := \alpha_1^{-1}\alpha_2$.

it suffices to check $d(\alpha_1(y_1), \alpha_2(y_2))^2 = d(y_1, y_2)^2 + |\alpha|^2$.

assume α nontrivial. $\Rightarrow X = Y' \times \mathbb{R}$.

assume $x_0 = (x_0, 0) \Rightarrow Y \subseteq Y' \times \{0\}$. $y_i = (y_i, 0)$, $i=1,2$.

$$\Rightarrow d(\alpha_1(y_1), \alpha_2(y_2))^2 = d(y_1, \alpha(y_2))^2 = d((y_1, 0), (y_2, |\alpha|))^2 = \dots$$

$V \in \text{Isom}(X)$, α Clifford $\Rightarrow V\alpha V^{-1}$ Clifford

V send axis of α through x to axis of $V\alpha V^{-1}$ through $V(x)$.

Hx is union of all axes of Clifford translation through x .

$$\Rightarrow V(H \cdot x) = H \cdot V(x)$$

$\Rightarrow$ by I.5.3(4), V preserve splitting. $\square$

Lemma 6.16: X complete CAT(0),

every geodesic can be extended to a geodesic line.

Γ acts cocompactly by isometries on X ,

$\alpha \in \text{Isom}(X)$ commutes with Γ ,

then α is a Clifford translation.

μ : $d\alpha$ is Γ -invariant

Γ acts cocompactly $\Rightarrow d\alpha$ is bounded.

$\forall x, y \in X, x \neq y, \exists$ geodesic segment c connecting x, y .

can be extended to a geodesic line $C: \mathbb{R} \rightarrow X$.

$t \mapsto d(C(t), \alpha(C(t)))$ is convex and bounded $\Rightarrow$ constant.

$\Rightarrow d_X(x) = d_X(y), \forall x, y \in X$

□

6.4 Group of Isometries of a compact metric space of non-positive curvature.

Recall: Prop I. 8.7: $\forall$ compact metric space, $\text{Isom}(X)$ equipped with metric:

$d(\alpha, \alpha') = \sup_{y \in Y} d(\alpha(y), \alpha'(y))$ is a compact topological group.

Def 5.7: geodesic metric space X is said to have the geodesic extension property, if $\forall$ local geodesic $C: [a, b] \rightarrow X, a \neq b$,

$\exists \varepsilon > 0, \exists C': [a, b + \varepsilon] \rightarrow X, C'|_{[a, b]} = C$.

Thm 6.17: $\forall$ compact connected metric space of non-positive curvature.

$Y = \tilde{P} \setminus X$, where $X \text{ CAT}(0)$. P acts properly and freely on X by isometries

$(X = \tilde{Y}, P = \pi_1(Y))$. Suppose Y has geodesic extension property.

Then: (1) $\text{Isom}(Y)$, equipped with metric $d(\alpha, \alpha') = \sup_{y \in Y} d(\alpha(y), \alpha'(y))$,

is a compact topological group, with finite connected components.

(2) connected component of identity in $\text{Isom}(Y)$ is isomorphic to a torus,

which is H_0/Λ , $H_0 \leq H$, Λ : lattice of H_0 .

where H is subspace of Clifford translation of X .

(3) $H_0 \cong C(P)$, centralizer of P in $\text{Isom}(X)$.

$\Lambda = C(\bar{P}) \cap \bar{P}$ is the center of $\bar{P}$.

(4) If the center of $\bar{P} = \pi_1(Y)$ is trivial, then $\text{Isom}(Y)$ is a finite group.

In general, the center of $\bar{P}$ is a free abelian group

whose rank $\leq$ maximal dimension of flat subspaces of X .

pf: Lemma 6.16 $\Rightarrow C(\bar{P})$ consists entirely of Clifford translations.

Now try to describe the component of identity in $\text{Isom}(Y)$

covering $p: X \rightarrow Y$, Y compact $\Rightarrow \exists \varepsilon > 0$, $\forall \bar{B}(x, \varepsilon) \subset X$, $p|_{\bar{B}(x, \varepsilon)}$ is a isometry.

If $\bar{v} \in \text{Isom}(Y)$, $d(\bar{v}, \text{id}) = a \leq \varepsilon$,

we have v , a lift of $\bar{v}$, which sent $x \in X$ to the unique point in $\bar{B}(x, \varepsilon)$,

whose image under p is $\bar{v}p(x)$.

$\Rightarrow v \in \text{Isom}(X)$, v commutes with every element of $\bar{P}$

$\Rightarrow v$ Clifford translation $\Rightarrow d(x, vx) = d(p(x), \bar{v}p(x)) = a$

By Prop 6.15: Clifford translations which centralize $\bar{P}$ form
a Hilbert subspace $H_0 \leq H$.

X locally compact $\Rightarrow H$ finite dimensional.

Homomorphism $P_*: C(\bar{P}) = H_0 \rightarrow \text{Isom}(Y)$,

sends $B(0, \varepsilon)$ in H_0 isometrically onto $B(\text{id}, \varepsilon)$ in $\text{Isom}(Y)$.

$\Rightarrow$ identity has a connected open neighborhood in $\text{Isom}(Y)$

$\text{Isom}(Y)$ is a compact topological group

$\Rightarrow$ locally connected $\Rightarrow$ finite connected components $\Rightarrow (1)$

Note that P_* is a covering map from H_0 onto the connected
component of identity in $\text{Isom}(Y)$.

$\Rightarrow \ker \beta_* = C(P) \cap P \triangleq \Lambda$ is a lattice in $H_0 \Rightarrow (2) \& (3)$.

(4) Λ trivial $\Rightarrow H_0 \cong \text{Isom}(Y)^\circ$ compact

H_0 vector space $\Rightarrow \text{Isom}(Y)^\circ = \{\text{id}\} \Rightarrow \text{Isom}(Y)$ finite

$\text{rank}(\Lambda) = \dim H_0$, Prop 6.15 $\Rightarrow H_0$ is isometrically embedded in X $\square$

Lemma 6.20: X complete CAT(0) with geodesic extension property.

group Γ acts cocompactly on X by isometries.

then X contains no Γ -invariant, closed, convex subsets other than $X, \emptyset$.

Proof: Lemma 6.16 can be view as a special case of Lemma 6.20.

pf: let C be non-empty Γ -invariant, closed, convex subset of X .

π : projection of X onto C .

Γ -invariant, cocompact $\Rightarrow \exists K > 0, \forall x \in X, d(x, C) \leq K$.

If $\exists x \in X \setminus C$, extend geodesic $[\pi(x), x]$ to a geodesic ray

$c: [0, +\infty) \rightarrow X$, s.t. $d(c(t), C) = t, \forall t \Rightarrow$ contradiction. $\square$